

ASYMPTOTIC CORRELATIONS FOR GAUSSIAN AND WISHART MATRICES WITH EXTERNAL SOURCE

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ABSTRACT. We consider ensembles of Gaussian (Hermite) and Wishart (Laguerre) $N \times N$ hermitian matrices. We study the effect of finite rank perturbations of these ensembles by a source term. The rank r of the perturbation corresponds to the number of non-null eigenvalues of the source matrix. In the perturbed ensembles, the correlation functions can be written in terms of kernels. We show that for all N , the difference between the perturbed and the unperturbed kernels is a degenerate kernel of size r which depends on multiple Hermite or Laguerre functions. We also compute asymptotic formulas for the multiple Laguerre functions kernels in terms multiple Bessel (resp. Airy) functions. This leads to the large N limiting kernels at the hard (resp. soft) edge of the spectrum of the perturbed Laguerre ensemble. Similar results are obtained in the Hermite case.

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1. INTRODUCTION

We study two specific ensembles of $N \times N$ random hermitian matrices \mathbf{X} in which the probability distribution function (p.d.f.) takes the general form

$$P(\mathbf{X}|\mathbf{A}) \propto e^{-\text{Tr}V(\mathbf{X}) - \text{Tr}\mathbf{A}\mathbf{X}}, \quad (1.1)$$

where \mathbf{A} is a given $N \times N$ hermitian matrices and $V(\mathbf{X})$ is a matrix-valued function of \mathbf{X} . These matrix ensembles have been considered by many authors, notably Brézin and Hikami [9], Guhr and Wettig [20, 21], Zinn-Justin [33]; and more recently, Baik, Ben Arous and P\'ech\'e [4, 3, 29], El Karoui [12], Bleher and Kuijlaars [5, 6], Imamura and Sasamoto [23].

Using the celebrated Harish-Chandra [22] (or Itzykson-Zuber [24]) formula,

$$\int_{\mathbf{U} \in U(N)} e^{\text{Tr}(\mathbf{U}\mathbf{A}\mathbf{U}^\dagger \mathbf{B})} (d\mathbf{U}) = \prod_{j=1}^N \Gamma(j) \frac{\det[e^{a_i b_j}]_{i,j=1,\dots,N}}{\det[a_i^{j-1}]_{i,j=1,\dots,N} \det[b_i^{j-1}]_{i,j=1,\dots,N}} \quad (1.2)$$

where $(d\mathbf{U})$ stands for the normalized Haar measure, one readily shows that the eigenvalue joint p.d.f. is

$$P_N(\mathbf{x}|\mathbf{a}) = \frac{1}{Z_N} \det[e^{-a_j x_k}]_{j,k=1}^N \prod_{i=1}^N e^{-V(x_i)} \prod_{1 \leq j < k \leq N} \frac{x_k - x_j}{a_k - a_j}, \quad (1.3)$$

where $\mathbf{x} = (x_1, \dots, x_N)$ denotes the random eigenvalues of \mathbf{X} while $\mathbf{a} = (a_1, \dots, a_N)$ denotes the fixed eigenvalues of \mathbf{A} . The quantity Z_N is the normalization. We suppose that $x_1 \geq \dots \geq x_N$ and that $x_j \in J \subseteq \mathbb{R}$. An essential feature of (1.3) is that it is of the form

$$P_N(\mathbf{x}|\mathbf{a}) = \frac{1}{Z_N} \prod_{1 \leq j < k \leq N} \frac{1}{a_k - a_j} \det[\xi_j(x_k)]_{j,k=1}^N \det[\eta_j(x_k)]_{j,k=1}^N \quad (1.4)$$

(set $\xi_j(x) = x^{j-1} e^{-a_j x - V(x)}$, $\eta_j(x) = x^{j-1}$, and make use of the Vandermonde determinant evaluation in the latter). We are thus dealing with determinantal point processes, which can be studied as biorthogonal ensembles in the sense of Borodin [8].

In this article, we consider the large N behavior of the correlations in two biorthogonal ensembles having an eigenvalue p.d.f. of the form (1.3),

- 1) $V(x) = x^2$ $J = (-\infty, \infty)$ Gaussian (Hermite) Unitary Ensemble;
 - 2) $V(x) = x - \alpha \ln(x)$ $J = (0, \infty)$ Wishart (Laguerre) Unitary Ensemble.
- (1.5)

To simplify subsequent integral representations, α will be taken to be a non-negative integer; however we expect our final formulas, appropriately interpreted, to be valid for general real $\alpha > -1$.

Both the choices in (1.5) can be realized by matrices with Gaussian entries. Consider case 1). Let \mathbf{A} be an $N \times N$ Hermitian matrix with distinct eigenvalues a_1, \dots, a_N . Let \mathbf{Y} be an element of the GUE, and thus be an $N \times N$ complex Hermitian matrix with probability density function proportional to $e^{-\text{Tr} \mathbf{Y}^2}$. With t a positive scalar, it is a well known consequence of (1.2) (see e.g. [17]) that the eigenvalue p.d.f. of

$$-\mathbf{A} + \sqrt{t}\mathbf{Y} \quad (1.6)$$

is equal to

$$\frac{(-1)^{N(N-1)/2}}{N!(2\pi t)^{N/2}} \det[e^{-(\lambda_i + a_j)^2/2t}]_{i,j=1,\dots,N} \frac{\det[\lambda_i^{j-1}]_{i,j=1,\dots,N}}{\det[a_i^{j-1}]_{i,j=1,\dots,N}}. \quad (1.7)$$

Simple manipulation and use of the Vandermonde determinant identity reduces (1.7) in the case $t = 1$ to (1.3) with $V(x) = x^2$.

Now consider case 2). Let \mathbf{W} be a $n \times N$ ($n \geq N$) complex Gaussian matrix with real and imaginary parts having variance $1/2$, mean 0, and let \mathbf{B} be an $N \times N$ positive definite Hermitian matrix with distinct eigenvalues b_1, \dots, b_N . Again it is a straightforward consequence of (1.2) [4, 7, 19, 30] that the eigenvalue p.d.f. of

$$\mathbf{B}^{-1/2} \mathbf{W}^\dagger \mathbf{W} \mathbf{B}^{-1/2} \quad (1.8)$$

is equal to

$$\frac{(-1)^{N(N-1)/2}}{N!} \prod_{j=1}^N \frac{b_j^N \lambda_j^{n-N}}{(n-N+j-1)!} \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - \lambda_j)}{(b_i - b_j)} \det[e^{-b_i \lambda_j}]_{i,j=1,\dots,N} \quad (1.9)$$

and thus with $b_i = 1 + a_i$, $\lambda_j = x_j$, is proportional to (1.3) with $V(x) = x - (n - N) \log x$.

A specialization of (1.8) of interest in mathematical statistics [25] is to take $\mathbf{B} = \text{diag}(b_1, \dots, b_r, 1, \dots, 1)$. Then the matrix $\mathbf{X} := \mathbf{W}^\dagger \mathbf{B}^{-1/2}$ has columns j , ($j = 1, \dots, r$) with variances $1/2\sqrt{b_j}$, while the entries in the remaining columns all have variance $1/2$. Regarding \mathbf{X} as a data matrix, this corresponds to so called spiked data. It was shown in [4] that with the scaled variables s_j and X_j specified by

$$b_j = \frac{1}{2} + \frac{1}{2(2N)^{1/3}} s_j \quad (j = 1, \dots, r) \quad x_j = 4N + 2(2N)^{1/3} X_j \quad (j = 1, \dots, k) \quad (1.10)$$

the scaled k -point correlation

$$\left(\frac{1}{2(2N)^{1/3}} \right)^k \rho_{(k)}(x_1, \dots, x_k) \quad (1.11)$$

tends in the limit $N \rightarrow \infty$, $n/N \rightarrow 1$ to

$$\rho_{(k)}^{\text{soft}\{s_j\}}(X_1, \dots, X_k) := \det \left[\mathcal{K}^{\text{soft}}(X_\mu, X_\nu) \right]_{\mu, \nu=1, \dots, k} \quad (1.12)$$

where

$$\mathcal{K}^{\text{soft}}(X, Y) = K^{\text{soft}}(X, Y) + \sum_{i=1}^r \widetilde{\text{Ai}}^{(i)}(X) \text{Ai}^{(i)}(Y) \quad (1.13)$$

with

$$K^{\text{soft}}(X, Y) = \frac{\text{Ai}(X) \text{Ai}'(Y) - \text{Ai}'(X) \text{Ai}(Y)}{X - Y}, \quad (1.14)$$

$$\widetilde{\text{Ai}}^{(i)}(x) = \int_{\mathcal{A}_{\{s_1, \dots, s_i\}}} \frac{dv}{2\pi i} e^{-xv + v^3/3} \prod_{k=1}^i (v - s_k)^{-1}, \quad (1.15)$$

$$\text{Ai}^{(i)}(x) = (-1)^i \int_{\mathcal{A}} \frac{dv}{2\pi i} e^{-xv + v^3/3} \prod_{k=1}^{i-1} (v + s_k). \quad (1.16)$$

See Fig. 2 for the definition of the contours $\mathcal{A}_{\{s_1, \dots, s_i\}}$ and \mathcal{A} . (The same limit (1.12) was found for $N \rightarrow \infty$ with $n/N \rightarrow \gamma^2$, $\gamma^2 \in \mathbb{R}^+$, provided the scales in (1.10) are suitably modified.) The significance of the scale (1.10) is that it centres the coordinates about the neighbourhood of the largest eigenvalue, and scales these eigenvalues so that their spacing is of order unity. Because the eigenvalue density is not strictly zero beyond the expected position of the largest eigenvalue, this is referred to as the soft edge, and thus the reason for the superscript ‘soft’ in (1.12) and (1.13).

Subsequent to the derivation of (1.12) as a limiting correlation at the soft edge for matrices (1.8) with \mathbf{B} a rank r perturbation of the identity, it was shown in [29] that (1.12) is also the limiting correlation at the soft edge for matrices (1.6) with \mathbf{A} a rank r perturbation of the zero matrix. Explicitly, if in (1.6) we take $t = 1$ and $\mathbf{A} = \text{diag}(a_1, \dots, a_r, 0, \dots, 0)$, with $a_1 = \dots = a_r = 1$, and introduce the scaled variables $x_j = \sqrt{2}N^{1/6}(\lambda_j - \sqrt{2N})$ ($j = 1, \dots, k$), then in the limit $N \rightarrow \infty$ the scaled k -point correlation

$$(\sqrt{2}N^{1/6})^k \rho_{(k)}(x_1, \dots, x_k)$$

tends to (1.12) with $s_1 = \dots = s_r = 0$.

In both the studies [4], [29] the structure (1.13) appears after the asymptotic analysis of the respective double contour representation of the general finite N , (μ, ν) element in (1.13) (which is called the correlation kernel). It is an objective of this to identify the structure analogous to (1.13) in the finite N p.d.f.s (1.7), (1.9). Moreover for (1.9) we identify a hard edge scaling (a scaling of the smallest eigenvalues in (1.9)) to the limiting k -point correlation

$$\rho_{(k)}^{\text{hard}\{h_j\}}(X_1, \dots, X_k) := \det \left[\mathcal{K}^{\text{hard}}(X_\mu, X_\nu) \right]_{\mu, \nu=1, \dots, k} \quad (1.17)$$

where

$$\mathcal{K}^{\text{hard}}(X, Y) = \left(\frac{Y}{X} \right)^{(\alpha+r)/2} K_{\alpha+r}^{\text{hard}}(X, Y) + \sum_{i=1}^r \tilde{J}^{(i)}(X) J^{(i)}(Y) \quad (1.18)$$

with

$$K_\alpha^{\text{hard}}(X, Y) = \frac{J_\alpha(\sqrt{X})\sqrt{Y}J'_\alpha(\sqrt{Y}) - \sqrt{X}J'_\alpha(\sqrt{X})J_\alpha(\sqrt{Y})}{2(X - Y)}, \quad (1.19)$$

$$\tilde{J}^{(i)}(x) = \int_{\mathcal{C}_{\{0, h_1, \dots, h_i\}}} \frac{dz}{2\pi i} \frac{e^{-xz+1/4z} z^{\alpha+r}}{\prod_{k=1}^i (z - h_k)}, \quad (1.20)$$

$$J^{(i)}(x) = \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi i} \frac{e^{xw-1/4w} \prod_{k=1}^{i-1} (w - h_k)}{w^{\alpha+r}}. \quad (1.21)$$

Here $\mathcal{C}_{\{0, h_1, \dots, h_i\}}$ is a simple closed anticlockwise contour encircling the points h_i and the origin, while $\mathcal{C}_{\{0\}}$ is a simple closed anticlockwise contour encircling the origin (see Fig. 1).

The correlation functions obtained here are multiparameter generalizations of the correlations in the Hermite and Laguerre unitary ensembles. The latter quantities are reviewed in the next section. In §3, we present some elements of the theory of multiple orthogonal functions, which provides the natural setting for studying the matrix ensembles with p.d.f. of the form (1.3). We show in §4 that k -point correlation function can be written as $\det[K(x_i, x_j)]_{i,j=1, \dots, k}$ for a given kernel $K(x_i, x_j)$. The finite rank, r say, perturbations of the Hermite and Laguerre ensembles are considered in §5. We show that even for matrix ensembles of finite size N the kernel takes the form exhibited by (1.13), (1.18). Explicitly $K_N(x, y|\mathbf{a}) = K_{N-r}(x, y|\mathbf{0}) + \sum_{i=1}^r \tilde{f}_i(x)f_i(y)$, where \tilde{f}_i and f_i denote multiple functions that generalize the Hermite or the Laguerre polynomials. In §6, we consider the large N limit and obtain the scaled kernels (1.13), (1.18). This is achieved by showing that the multiple Laguerre functions tend to multiple Bessel functions at the hard edge, while both the multiple Laguerre and Hermite functions tend to the same multiple Airy functions at the soft edge. The article ends with a short discussion.

2. NULL CASE

The null case corresponds to ensembles of random matrices with unitary symmetry (see for instance [15, 27]). It is defined by

$$\mathbf{a} \longrightarrow \mathbf{0} = \{0, \dots, 0\} \quad (2.1)$$

in (1.3). In such a situation, $\prod_{i < j} (a_j - a_i) \det [e^{-a_k x_l}]_{k,l}$ tends to $\prod_{i < j} (x_j - x_i)$, and the k -point correlation function can be written as a $k \times k$ determinant.

In order to be more concrete, let us introduce $p_n(x) = x^n + c_{n-1}x^{n-1} + \dots$, an orthogonal polynomial with respect to the weight $w(x) := e^{-V(x)}$ supported on the real interval J ; that is,

$$\int_J dx w(x) p_j(x) p_k(x) = \|p_j\|^2 \delta_{j,k}. \quad (2.2)$$

Let also (Christoffel-Darboux formula [31])

$$\begin{aligned} K_N(x, y) &:= \sqrt{w(x)w(y)} \sum_{n=0}^{N-1} \frac{p_n(x)p_n(y)}{\|p_n\|^2} \\ &= \frac{\sqrt{w(x)w(y)}}{\|p_{N-1}\|^2} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x - y} \end{aligned} \quad (2.3)$$

be the reproducing kernel on the interval J . It satisfies

$$\int_J dy K_N(x, y) K_N(y, z) = K_N(x, z), \quad \int_J dx K_N(x, x) = N \quad (2.4)$$

and

$$P_N(\mathbf{x}) := \frac{1}{Z_N} \prod_{i=1}^N w(x_i) \prod_{1 \leq j < k \leq N} (x_k - x_j)^2 = \det [K_N(x_i, x_j)]_{i,j=1}^N. \quad (2.5)$$

These properties together with the Laplace expansion of the determinant imply that the k -point correlation function can be expressed as a $k \times k$ determinant of the kernel,

$$\begin{aligned} \rho_{(k)}(x_1, \dots, x_k) &:= \frac{N!}{(N-k)!} \int_J dx_{k+1} \cdots \int_J dx_N P_N(x_1, \dots, x_N) \\ &= \det [K_N(x_i, x_j)]_{i,j=1}^k \end{aligned} \quad (2.6)$$

For the Hermite and Laguerre ensembles, the monic polynomials of degree n can respectively be expressed in terms of the Hermite and the Laguerre polynomials,

$$\begin{aligned} 1) \quad p_n(x) &= 2^{-n} H_n(x) & \|p_n\|^2 &= \sqrt{\pi} 2^{-n} n! & \text{if } J &= (-\infty, \infty), \\ 2) \quad p_n(x) &= (-1)^n n! L_n^\alpha(x) & \|p_n\|^2 &= n! (n + \alpha)! & \text{if } J &= (0, \infty). \end{aligned} \quad (2.7)$$

In the Hermite case, we thus have

$$K_N(x, y) = \frac{e^{-(x^2+y^2)/2}}{2^N \sqrt{\pi} (N-1)!} \frac{H_N(x) H_{N-1}(y) - H_{N-1}(x) H_N(y)}{x - y}, \quad (2.8)$$

while we have, in the Laguerre case,

$$K_N^\alpha(x, y) = -\frac{N!}{(N + \alpha - 1)!} (xy)^{\alpha/2} e^{-(x+y)/2} \frac{L_N^\alpha(x) L_{N-1}^\alpha(y) - L_{N-1}^\alpha(x) L_N^\alpha(y)}{x - y}. \quad (2.9)$$

Of particular interest are the scaled kernels [13], relating to the edge of the spectrum, which are defined by

$$K^{\text{scaled}}(X, Y) := \lim_{N \rightarrow \infty} B(N) K_N(A(N) + B(N)X, A(N) + B(N)Y), \quad (2.10)$$

where ‘scaled’ stands for ‘hard’ or ‘soft’. When the functions A and B are suitably chosen, one gets kernels that are independent of N and that can be expressed in terms of simple functions. In the Hermite ensemble, there is only one distinct edge scaling,

$$x = A(N) + B(N)X = \sqrt{2N} + X/\sqrt{2N^{1/3}}, \quad \text{soft edge.} \quad (2.11)$$

Two edge scaling are possible for the Laguerre ensemble,

$$x = A(N) + B(N)X = \begin{cases} X/4N, & \text{hard edge,} \\ 4N + 2(2N)^{1/3}X, & \text{soft edge.} \end{cases} \quad (2.12)$$

By substituting the corresponding scaling of Eqs. (2.11) and (2.12) into Eq. (2.10) the scaled kernels (1.14), (1.19) result.

3. MULTIPLE BIORTHOGONAL POLYNOMIALS

3.1. Generalities. We introduce the multiple Hermite and Laguerre polynomials [1, 5, 7]. Note that we use slightly different notations and definitions than those of the latter references in order to simplify the comparison between our asymptotic formulas and the well known formulas as revised above in the null case.

Let us suppose that within the set

$$\mathbf{a} = \{a_1, \dots, a_N\}, \quad (3.1)$$

only D eigenvalues are distinct. Then we may write

$$\mathbf{a} = \mathbf{b}^{\mathbf{m}} := \{b_1^{m_1}, \dots, b_D^{m_D}\}, \quad (3.2)$$

which means there are m_i eigenvalues in \mathbf{a} that are equal to b_i ($i = 1, \dots, D$). Let also $|\mathbf{m}| := \sum_{i=1}^D m_i = N$; that is, \mathbf{m} is a composition of non-negative integers, of weight N and of fixed length D .

The i th multiple orthogonal polynomials of type I associated to a given set of parameters \mathbf{b} , denoted by $Q_{\mathbf{m},i}(x)$ or $Q_{\mathbf{m},i}(x|\mathbf{b})$ (but usually, the dependence on \mathbf{b} is kept implicit), is a polynomial in x of degree $m_i - 1$. It is used to build the function $Q_{\mathbf{m}}(x) := \sum_{i=1}^D e^{-b_i x} Q_{\mathbf{m},i}(x)$ such that

$$\int_J dx w(x) x^j Q_{\mathbf{m}}(x) = \begin{cases} 0, & j = 0, \dots, |\mathbf{m}| - 2, \\ 1, & j = |\mathbf{m}| - 1. \end{cases} \quad (3.3)$$

The multiple orthogonal polynomial of type II and of degree $|\mathbf{m}|$ is denoted by $P_{\mathbf{m}}(x)$ or $P_{\mathbf{m}}(x|\mathbf{b})$. It satisfies

$$\int_J dx w(x) e^{-b_i x} x^j P_{\mathbf{m}}(x) = 0, \quad j = 0, \dots, m_i - 1, \quad (3.4)$$

where $i = 1, \dots, D$. Combining the two previous equations, we see that

$$\int_J dx w(x) P_{\mathbf{m}}(x) Q_{\mathbf{n}}(x) = \begin{cases} 0, & |\mathbf{m}| < |\mathbf{n}| - 1, \\ 1, & |\mathbf{m}| = |\mathbf{n}| - 1, \\ 0, & m_i > n_i - 1 \quad \forall i. \end{cases} \quad (3.5)$$

In other words, for a given set of parameters \mathbf{b} and a given composition \mathbf{m} of weight N , the polynomials of type I and II allow us to build a family of biorthogonal functions,

$$\int_J dx w(x) p_i(x) q_j(x) = \delta_{i,j} \quad \text{for all } i, j = 0, \dots, N-1 \quad (3.6)$$

by choosing

$$p_i(x) = P_{\boldsymbol{\mu}_i}(x), \quad q_i(x) = Q_{\boldsymbol{\mu}_{i+1}}(x), \quad (3.7)$$

and if for instance, the compositions $\boldsymbol{\mu}_i$ are given by

$$\begin{pmatrix} \boldsymbol{\mu}_N \\ \boldsymbol{\mu}_{N-1} \\ \boldsymbol{\mu}_{N-2} \\ \boldsymbol{\mu}_{N-3} \\ \vdots \\ \boldsymbol{\mu}_0 \end{pmatrix} = \begin{pmatrix} m_1 & m_2 & m_3 & \dots & m_D \\ m_1 - 1 & m_2 & m_3 & \dots & m_D \\ m_1 - 1 & m_2 - 1 & m_3 & \dots & m_D \\ m_1 - 1 & m_2 - 1 & m_3 - 1 & \dots & m_D \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (3.8)$$

where it is understood that $m_i \geq m_{i+1}$.

3.2. Multiple Laguerre functions. The multiple Laguerre function of type I is [7, cf. Eq. (3.10)]

$$\tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(x) = \sum_{i=1}^D e^{-b_i x} \tilde{\mathcal{L}}_{\mathbf{m},i}^{\alpha}(x) := \int_{\mathcal{C}_{\mathbf{b}}^{\{-1\}}} \frac{dz}{2\pi i} \frac{e^{-xz} (1+z)^{|\mathbf{m}|+\alpha-1}}{\prod_{i=1}^D (z-b_i)^{m_i}}, \quad \alpha \in \mathbb{Z}, \quad (3.9)$$

where $\mathcal{C}_{\mathbf{b}}^{\{-1\}} = \mathcal{C}_{\{b_1, \dots, b_D\}}^{\{-1\}}$ denotes a closed contour which encircles positively b_1, \dots, b_D but not the point -1 (See Fig. 1). Note that encircling the point -1 is forbidden only when $|\mathbf{m}| + \alpha - 1$ is negative. This means in particular that for $\alpha > -1$, the contour $\mathcal{C}_{\mathbf{b}}^{\{-1\}}$ can be replaced by $\mathcal{C}_{\mathbf{b}}$. Obviously, the i th polynomial $\tilde{\mathcal{L}}_{\mathbf{m},i}^{\alpha}(x)$ has degree $m_i - 1$. The multiple Laguerre polynomial of type II has degree $|\mathbf{m}|$; it can be defined by [7, cf. Eq. (3.5)]

$$\mathcal{L}_{\mathbf{m}}^{\alpha}(x) := \frac{(|\mathbf{m}| + \alpha)!}{|\mathbf{m}|!} x^{-\alpha} \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi i} \frac{e^{xw}}{w^{|\mathbf{m}|+\alpha+1}} \prod_{i=1}^D (w-1-b_i)^{m_i}, \quad \alpha \in \mathbb{Z}, \quad (3.10)$$

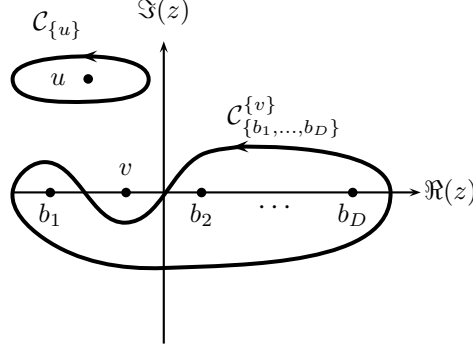
where $\mathcal{C}_{\{0\}}$ is a counterclockwise contour around the origin. Our definitions, modifying those of [7] are motivated in part by the first of the following results, while the key orthogonality type results remain essentially unchanged.

Proposition 1. *Let $L_n^{\alpha}(x)$ denote the Laguerre polynomial of degree n . Set α a non-negative integer and $b_1, \dots, b_D > -1$. Then the multiple Laguerre functions satisfy*

$$\lim_{\mathbf{b} \rightarrow \mathbf{0}} \tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(x) = L_{|\mathbf{m}|-1}^{\alpha}(x) \quad \text{and} \quad \lim_{\mathbf{b} \rightarrow \mathbf{0}} \mathcal{L}_{\mathbf{m}}^{\alpha}(x) = L_{|\mathbf{m}|}^{\alpha}(x).$$

Moreover [7, Thm. 3.2]

$$\int_0^{\infty} dx e^{-x} x^{\alpha+j} \tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(x) = \begin{cases} 0, & j = 0, \dots, |\mathbf{m}| - 2 \\ \frac{(-1)^{|\mathbf{m}|-1} (|\mathbf{m}| + \alpha - 1)!}{\prod_{i=1}^D (1+b_i)^{m_i}}, & j = |\mathbf{m}| - 1, \end{cases}$$

FIGURE 1. Contours $\mathcal{C}_{\{u\}}$ and $\mathcal{C}_{\{b_1, \dots, b_D\}}^{\{v\}}$ in the complex z -plane.

and [7, Thm. 3.1]

$$\int_0^\infty dx e^{-x-b_i x} x^{\alpha+j} \mathcal{L}_{\mathbf{m}}^\alpha(x) = 0 \quad \text{for } i = 1, \dots, D \quad \text{and } j = 0, \dots, m_i - 1.$$

Proof. One easily proves the two first properties by considering the integral representation of the Laguerre polynomials [31],

$$L_n^\alpha(x) = \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi i} \frac{e^{-xw}}{w^{n+1}} (1+w)^{n+\alpha} \quad (3.11)$$

and the readily verified formula

$$n!(-x)^{-\alpha} L_n^{-\alpha}(x) = (n-\alpha)! L_{n-\alpha}^\alpha(x), \quad (3.12)$$

both relations being valid for $\alpha \in \mathbb{Z}$. Then, using Eq. (3.9), we have

$$\begin{aligned} \int_0^\infty dx e^{-x} x^{\alpha+j} \tilde{\mathcal{L}}_{\mathbf{m}}^\alpha(x) &= \int_{\mathcal{C}_{\mathbf{b}}} \frac{dz}{2\pi i} \frac{(1+z)^{|\mathbf{m}|+\alpha-1}}{\prod_{i=1}^D (z-b_i)^{m_i}} \int_0^\infty dx e^{-x-xz} x^{\alpha+j} \\ &= (\alpha+j)! \int_{\mathcal{C}_{\mathbf{b}}^{\{-1\}}} \frac{dz}{2\pi i} \frac{(1+z)^{|\mathbf{m}|-j-2}}{\prod_{i=1}^D (z-b_i)^{m_i}} \end{aligned}$$

Note that, on the first line, we assumed $b_i > -1$ and $\Re(z) > -1$; this allowed us to do the x -integration. For $j = |\mathbf{m}| - 1$, we use the residue theorem and get

$$\int_0^\infty dx e^{-x} x^{\alpha+j} \tilde{\mathcal{L}}_{\mathbf{m}}^\alpha(x) = -(|\mathbf{m}| + \alpha - 1)! \text{Res}_{z=-1} \left[\prod_{i=1}^D (z-b_i)^{-m_i} \right] = -\frac{(|\mathbf{m}| + \alpha - 1)!}{\prod_{i=1}^D (-1-b_i)^{m_i}}$$

as desired. For $j < |\mathbf{m}| - 1$, we consider $(1+z)^{|\mathbf{m}|-j-2} \prod_{i=1}^D (z-b_i)^{-m_i} \rightarrow z^{-j-2}$ when $z \rightarrow \infty$. Thus, by virtue of the residue theorem,

$$\int_0^\infty dx e^{-x} x^{\alpha+j} \tilde{\mathcal{L}}_{\mathbf{m}}^\alpha(x) \propto \text{Res}_{z=\infty} [z^{-j-2}] = 0 \quad \text{when } -1 < j < |\mathbf{m}| - 1.$$

The same tricks are used to prove the second orthogonality property. Explicitly,

$$\begin{aligned}
\int_0^\infty dx e^{-x-b_i x} x^{\alpha+j} \mathcal{L}_{\mathbf{m}}^\alpha(x) &\propto \int_{\substack{\mathcal{C}_{\{0\}} \\ \Re(w) < 1+b_i}} \frac{dw}{2\pi i} \frac{\prod_{j=1}^D (w-1-b_j)^{m_j}}{w^{|\mathbf{m}|+\alpha+1}} \int_0^\infty dx e^{-x-b_i x+xw} x^j \\
&\propto \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi i} \frac{(w-1-b_i)^{-j-1} \prod_{j=1}^D (w-1-b_j)^{m_j}}{w^{|\mathbf{m}|+\alpha+1}} \\
&\propto \text{Res}_{w=\infty} [w^{-j-\alpha-2}] = 0.
\end{aligned}$$

when $-1 < j < m_i$ and $\alpha > -1$, as proposed. \square

Comparing with the general formalism of §3.1 we have

$$\begin{aligned}
w(x) &= x^\alpha e^{-x} \\
Q_{\mathbf{m}}(x) &= \frac{(-1)^{|\mathbf{m}|-1} \prod_{i=1}^D (1+b_i)^{m_i}}{(\mathbf{m} + \alpha - 1)!} \tilde{\mathcal{L}}_{\mathbf{m}}^\alpha(x) \\
P_{\mathbf{m}}(x) &= \mathcal{L}_{\mathbf{m}}^\alpha(x)
\end{aligned} \tag{3.13}$$

with corresponding biorthogonal functions specified by (3.7).

The multiple Laguerre functions also satisfy certain differential equations [1]. Indeed, if

$$\mathcal{D}_{t,x} f(x) := -e^{tx} \partial_x (e^{-tx} f(x)) = (t - \partial_x) f(x), \quad \text{with} \quad \partial_x = \frac{\partial}{\partial x}, \tag{3.14}$$

then

$$\mathcal{D}_{t,x} [e^{-xz}] = (z+t)e^{-xz}. \tag{3.15}$$

Consequently,

$$0 = \mathcal{D}_{-a_1,x} \cdots \mathcal{D}_{-a_{|\mathbf{m}|},x} [\tilde{\mathcal{L}}_{\mathbf{m}}^\alpha(x)] \tag{3.16}$$

and

$$x^\alpha \mathcal{L}_{\mathbf{m}}^\alpha(x) = \frac{(-1)^{|\mathbf{m}|}}{|\mathbf{m}|!} \mathcal{D}_{1+a_1,x} \cdots \mathcal{D}_{1+a_{|\mathbf{m}|},x} [x^{|\mathbf{m}|+\alpha}]. \tag{3.17}$$

When m_D parameters in \mathbf{a} are equal to zero, $a_{|\mathbf{m}|-m_D} = \dots = a_{|\mathbf{m}|} = 0$ say, after some simple manipulations we can show that the multiple Laguerre functions of type I and II can be respectively interpreted as anti-derivatives and derivatives of Laguerre polynomials. Explicitly, for $r = |\mathbf{m}| - m_D$, we see from the integral representations (3.9), (3.22) and the differentiation formula (3.15) that

$$L_{|\mathbf{m}|-r-1}^{\alpha+r} = \mathcal{D}_{-a_1,x} \cdots \mathcal{D}_{-a_r,x} [\tilde{\mathcal{L}}_{\mathbf{m}}^\alpha(x)] \tag{3.18}$$

while the integral representations (3.9), (3.22), the identity (3.15) and the differentiation formula (3.15) show that

$$x^\alpha \mathcal{L}_{\mathbf{m}}^\alpha(x) = (-1)^r \frac{(|\mathbf{m}| - r)!}{|\mathbf{m}|!} \mathcal{D}_{a_1,x} \cdots \mathcal{D}_{a_r,x} [x^{\alpha+r} L_{|\mathbf{m}|-r}^{\alpha+r}(x)]. \tag{3.19}$$

3.3. Multiple Hermite functions. The multiple Hermite polynomials of type I, written $\tilde{\mathcal{H}}_{\mathbf{m},i}(x)$, are obtained from the function [7, cf. (2.9)]

$$\tilde{\mathcal{H}}_{\mathbf{m}}(x) = \sum_{i=1}^D e^{-b_i x} \tilde{\mathcal{H}}_{\mathbf{m},i}(x) = (-2)^{|\mathbf{m}|-1} (|\mathbf{m}| - 1)! \int_{\mathcal{C}_{\mathbf{b}}} \frac{dv}{2\pi i} \frac{e^{-v^2/4 - xv}}{\prod_{i=1}^D (v - b_i)^{m_i}}, \quad (3.20)$$

where $\mathcal{C}_{\mathbf{b}}$ is a simple contour which encircles the points $\{b_1, \dots, b_D\}$ anticlockwise. The (non-monic) multiple Hermite polynomial of type II is given by [7, cf. (2.2)]

$$\mathcal{H}_{\mathbf{m}}(x) = (-1)^{|\mathbf{m}|} \sqrt{\pi} e^{x^2} \int_{-i\infty}^{i\infty} \frac{du}{2\pi i} e^{u^2/4 + xu} \prod_{i=1}^D (u - b_i)^{m_i}. \quad (3.21)$$

One can verify that the degrees of $\tilde{\mathcal{H}}_{\mathbf{m},i}(x)$ and $\mathcal{H}_{\mathbf{m}}(x)$ are respectively $m_i - 1$ and $|\mathbf{m}|$.

The multiple Hermite functions (polynomials) can be obtained as a limit of their Laguerre counterpart.

Proposition 2. *Let X be real numbers such that $|\sqrt{2\alpha}X| < \alpha$ with $\alpha \in \mathbb{N}$. Then, as $\alpha \rightarrow \infty$,*

$$(2\alpha)^{(1-|\mathbf{m}|)/2} \tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(\alpha + \sqrt{2\alpha}X) \Big|_{b_i \mapsto b_i/\sqrt{2\alpha}} = \frac{(-1)^{|\mathbf{m}|-1}}{2^{|\mathbf{m}|-1} (|\mathbf{m}| - 1)!} \tilde{\mathcal{H}}_{\mathbf{m}}(X) + O\left(\frac{1}{\sqrt{\alpha}}\right)$$

and

$$(2\alpha)^{-|\mathbf{m}|/2} \mathcal{L}_{\mathbf{m}}^{\alpha}(\alpha + \sqrt{2\alpha}X) \Big|_{b_i \mapsto b_i/\sqrt{2\alpha}} = \frac{(-1)^{|\mathbf{m}|}}{2^{|\mathbf{m}|} |\mathbf{m}|!} \mathcal{H}_{\mathbf{m}}(X) + O\left(\frac{1}{\sqrt{\alpha}}\right).$$

Proof. The proof of the first asymptotic formula is the simplest. From the definition of the type I Laguerre functions we have

$$(2\alpha)^{(1-|\mathbf{m}|)/2} \tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(\alpha + \sqrt{2\alpha}X) \Big|_{b_i \mapsto b_i/\sqrt{2\alpha}} = (2\alpha)^{(1-|\mathbf{m}|)/2} \int_{\mathcal{C}_{\mathbf{b}/\sqrt{2\alpha}}} \frac{dz}{2\pi i} \frac{e^{-\alpha z - \sqrt{2\alpha}Xz} (1+z)^{|\mathbf{m}|+\alpha-1}}{\prod_{i=1}^D (z - b_i/\sqrt{2\alpha})^{m_i}}.$$

By changing $z \mapsto z/\sqrt{2\alpha}$, we get

$$(2\alpha)^{(1-|\mathbf{m}|)/2} \tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(\alpha + \sqrt{2\alpha}X) \Big|_{b_i \mapsto b_i/\sqrt{2\alpha}} = \int_{\mathcal{C}_{\mathbf{b}}} \frac{dz}{2\pi i} \frac{e^{-Xz - \sqrt{\alpha/2}z} (1+z/\sqrt{2\alpha})^{\alpha+|\mathbf{m}|-1}}{\prod_{i=1}^D (z - b_i)^{m_i}}.$$

From the binomial development $(1 + z/\sqrt{2\alpha})^{\alpha+|\mathbf{m}|-1}$ and the Maclaurin expansion of $e^{-\sqrt{\alpha/2}z}$, we easily find that

$$\begin{aligned} e^{-\sqrt{\alpha/2}z} (1 + z/\sqrt{2\alpha})^{\alpha+|\mathbf{m}|-1} &= e^{-\sqrt{\alpha/2}z} (1 + z/\sqrt{2\alpha})^{\alpha} + O\left(\frac{1}{\sqrt{\alpha}}\right) \\ &= e^{-z^2/4} + O\left(\frac{1}{\sqrt{\alpha}}\right), \end{aligned}$$

when $\alpha \rightarrow \infty$. This immediately implies the first asymptotic formula.

The second asymptotic relation is established by steepest descents. The integral formula (3.10) shows

$$\mathcal{L}_{\mathbf{m}}^{\alpha}(\alpha + \sqrt{2\alpha}X) = \frac{(|\mathbf{m}| + \alpha)!}{|\mathbf{m}|!} (\alpha + \sqrt{2\alpha}X)^{-\alpha} \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi i} e^{\alpha f(w)} g(w),$$

where

$$f(w) = w - \ln w \quad \text{and} \quad g(w) = e^{-\alpha \ln(1+\sqrt{2/\alpha}X)} e^{\sqrt{2\alpha}Xw} \prod_{i=1}^D (w - 1 - b_i)^{m_i} w^{-|\mathbf{m}|-1}.$$

We point out that αw dominates $\sqrt{2\alpha}Xw$ by hypothesis. Thus, the function $f(w)$ determines the principal contribution to the integral when α becomes large. $f(w)$ possesses one simple saddle point at $w_0 = 1$; that is, $f'(w_0) = 0$ but $f''(w_0) = 1 \neq 0$. The steepest descent contour is fixed by the condition $(w - w_0)^2 f''(w_0) = \text{minimum}$, which means $\arg(w) = \pm\pi/2$. We consequently choose the contour \mathcal{C}_0 of w such that the latter variable starts at $1 - ic$ (c is real and positive), passes through the point $w = 1$, goes to $1 + ic$ and finally returns to the starting point after encircling that origin. We now set $v = \sqrt{2\alpha}(w - 1)$. When $\alpha \rightarrow \infty$

$$\begin{aligned} \alpha f(w) &= \alpha + v^2/4 + O\left(\frac{1}{\sqrt{\alpha}}\right); \\ (2\alpha)^{|\mathbf{m}|/2} g(w) &= e^{Xv+X^2} \prod_{i=1}^D (v - \sqrt{2\alpha}b_i) + O\left(\frac{1}{\sqrt{\alpha}}\right); \end{aligned}$$

and the steepest descent path in the complex v -plane becomes the interval $(-i\infty, i\infty)$ on the imaginary axis. We also know from Stirling's approximation that

$$(|\mathbf{m}| + \alpha)! = \sqrt{2\pi} e^{-\alpha} \alpha^{|\mathbf{m}|+\alpha+1/2} + O\left(\frac{1}{\alpha}\right).$$

Therefore, we have proved

$$(2\alpha)^{-|\mathbf{m}|/2} \mathcal{L}_{\mathbf{m}}^{\alpha}(\alpha + \sqrt{2\alpha}X) \Big|_{b_i \mapsto b_i/\sqrt{2\alpha}} = \frac{1}{2^{|\mathbf{m}|} |\mathbf{m}|!} \sqrt{\pi} e^{X^2} \int_{-i\infty}^{i\infty} \frac{dv}{2\pi i} e^{v^2/4+Xv} \prod_{i=1}^D (v - b_i)^{m_i},$$

which is the desired expression. \square

Proposition 3. *Let $H_n(x)$ be the Hermite polynomial of degree n . Then the multiple Hermite functions satisfy*

$$\lim_{\mathbf{b} \rightarrow \mathbf{0}} \tilde{\mathcal{H}}_{\mathbf{m}}(x) = H_{|\mathbf{m}|-1}(x) \quad \text{and} \quad \lim_{\mathbf{b} \rightarrow \mathbf{0}} \mathcal{H}_{\mathbf{m}}(x) = H_{|\mathbf{m}|}(x).$$

Moreover [7, Thm 2.3]

$$\int_{-\infty}^{\infty} dx e^{-x^2} x^j \tilde{\mathcal{H}}_{\mathbf{m}}(x) = \begin{cases} 0, & j = 0, \dots, |\mathbf{m}| - 2 \\ \sqrt{\pi} (|\mathbf{m}| - 1)!, & j = |\mathbf{m}| - 1, \end{cases}$$

and [7, Thm 2.1]

$$\int_{-\infty}^{\infty} dx e^{-x^2 - b_i x} x^j \mathcal{H}_{\mathbf{m}}(x) = 0 \quad \text{for } i = 1, \dots, D \quad \text{and } j = 0, \dots, m_i - 1.$$

Proof. The two first properties are straightforward consequences of well known integral representation of the Hermite polynomials [31]

$$H_n(x) = 2^n n! \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi i} \frac{e^{xw - w^2/4}}{w^{n+1}} = \sqrt{\pi} e^{x^2} \int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} w^n e^{-xw + w^2/4}. \quad (3.22)$$

We can prove the orthogonality relations by using Proposition 2 and substituting

$$e^{-\sqrt{2\alpha}x}(1 + \sqrt{\frac{2}{\alpha}}x)^\alpha = e^{-x^2} + O\left(\frac{1}{\sqrt{\alpha}}\right)$$

in the last equations of Proposition 1. \square

Suppose in (3.20) and (3.21) that $b_i = a_i$, $m_i = 1$ ($i = 1, \dots, r$) while $b_{r+1} = 0$, $m_{r+1} = |\mathbf{m}| - r$. Then we see from (3.15) and (3.22) that

$$\mathcal{D}_{-a_1, x} \cdots \mathcal{D}_{-a_r, x} \tilde{\mathcal{H}}_{\mathbf{m}}(x) = (-2)^r \frac{(|\mathbf{m}| - 1)!}{(|\mathbf{m}| - r - 1)!} H_{|\mathbf{m}| - r - 1}(x) \quad (3.23)$$

(cf. (3.18)) and

$$\mathcal{D}_{a_1, x} \cdots \mathcal{D}_{a_r, x} \left(e^{-x^2} H_{|\mathbf{m}| - r}(x) \right) = e^{-x^2} \tilde{\mathcal{H}}_{\mathbf{m}}(x) \quad (3.24)$$

4. NON-NULL CASE

4.1. Multiple Laguerre kernel. For the determinantal point processes (1.4), the method of biorthogonal polynomials [8] gives that the corresponding k -point correlation function has the determinant form

$$\rho_{(k)}(x_1, \dots, x_k) = \det[\mathcal{K}(x_i, x_j)]_{i,j=1,\dots,k} \quad (4.1)$$

for a certain $\mathcal{K}(x, y)$ referred to as the correlation kernel. The correlation kernel can always be written as a double sum over $\xi_i(x)\eta_j(y)$ weighted by a factor proportional to the inverse of the matrix of inner products $[(\xi_i, \eta_j)]$. We know from [4] that for the Laguerre case of (1.4) this double sum can be written as a double contour integral

$$\mathcal{K}_{\mathbf{m}}^\alpha(x, y) = \left(\frac{x}{y}\right)^{\alpha/2} e^{(y-x)/2} \int_{\mathcal{C}_{\mathbf{b}}^{\{-1\}}} \frac{dz}{2\pi i} \int_{\mathcal{C}_{\{-1\}}} \frac{dw}{2\pi i} \frac{e^{-xz+yw}}{w-z} \left(\frac{1+z}{1+w}\right)^{|\mathbf{m}|+\alpha} \prod_{i=1}^D \left(\frac{w-b_i}{z-b_i}\right)^{m_i}, \quad (4.2)$$

In the latter expression, α stands for a non-negative integer while $\mathcal{C}_{\mathbf{b}}^{\{-1\}}$ and $\mathcal{C}_{\{-1\}}$ stand for *non-intersecting* counterclockwise contours that encircle \mathbf{b} and -1 , respectively.

On the other hand, a determinantal point process

$$\frac{1}{Z_N} \prod_{l=1}^N w(x_l) \det[p_i(x_j)]_{i,j=1,\dots,N} \det[q_i(x_j)]_{i,j=1,\dots,N} \quad (4.3)$$

in which $\{p_i(x)\}$, $\{q_j(x)\}$ have the biorthogonality property, has for its correlation kernel the single sum

$$(w(x)w(y))^{1/2} \sum_{i=1}^N p_i(x)q_i(y) \quad (4.4)$$

(cf. the first line in (2.3)). Furthermore, the general theory of multiple orthogonal polynomials [11] tells us that with p_i, q_j as in (3.7), (3.8) can be summed according to a generalization of the Christoffel-Darboux formula in (2.3). In the Laguerre case this summation has been made explicit in [7]. We revise this latter formula, and make explicit the form of (4.4) in the following.

Proposition 4. Let $\mathbf{e}_i = (\overbrace{0, \dots, 0}^{i-1}, 1, 0, \dots)$. The correlation kernel (4.2) has the summed series form

$$\mathcal{K}_{\mathbf{m}}^{\alpha}(x, y) = \frac{|\mathbf{m}|!}{(|\mathbf{m}| + \alpha - 1)!} \frac{(xy)^{\alpha/2} e^{-(x+y)/2}}{x - y} \left(\tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(x) \mathcal{L}_{\mathbf{m}}^{\alpha}(y) - \sum_{i=1}^D \frac{m_i}{|\mathbf{m}|} \tilde{\mathcal{L}}_{\mathbf{m}+\mathbf{e}_i}^{\alpha}(x) \mathcal{L}_{\mathbf{m}-\mathbf{e}_i}^{\alpha}(y) \right)$$

and the single sum form

$$\mathcal{K}_{\mathbf{m}}^{\alpha}(x, y) = (xy)^{\alpha/2} e^{-(x+y)/2} \sum_{i=1}^D \sum_{j=1}^{m_i} \frac{(|\mathbf{m}| - j)!}{(|\mathbf{m}| + \alpha - j)!} e^{-b_i x} \tilde{\mathcal{L}}_{\mathbf{m}-(j-1)\mathbf{e}_i}^{\alpha}(x) \mathcal{L}_{\mathbf{m}-j\mathbf{e}_i}^{\alpha}(y).$$

Proof. First, we effectively multiply Eq. (4.2) by $(x - y)/(x - y)$ by writing

$$\begin{aligned} \mathcal{K}_{\mathbf{m}}^{\alpha}(x, y) &= \left(\frac{x}{y} \right)^{\alpha/2} \frac{e^{(y-x)/2}}{x - y} \\ &\times \int_{\mathcal{C}_{\mathbf{b}}^{\{-1\}}} \frac{dz}{2\pi i} \int_{\mathcal{C}_{\{-1\}}} \frac{dw}{2\pi i} \frac{1}{w - z} \left(\frac{1 + z}{1 + w} \right)^{N+\alpha} \prod_{i=1}^N \left(\frac{w - a_i}{z - a_i} \right) \left(-\frac{\partial}{\partial z} - \frac{\partial}{\partial w} \right) (e^{-xz+yw}), \end{aligned}$$

where use has been made of the relation between $\{b_i\}$ and $\{a_i\}$ noted in §3.1. Integrating by parts gives

$$\begin{aligned} \mathcal{K}_{\mathbf{m}}^{\alpha}(x, y) &= \left(\frac{x}{y} \right)^{\alpha/2} \frac{e^{(y-x)/2}}{x - y} \\ &\times \int_{\mathcal{C}_{\mathbf{b}}^{\{-1\}}} \frac{dz}{2\pi i} \int_{\mathcal{C}_{\{-1\}}} \frac{dw}{2\pi i} e^{-xz+yw} \left(\frac{1 + z}{1 + w} \right)^{N+\alpha} \prod_{i=1}^N \left(\frac{w - a_i}{z - a_i} \right) \left(\frac{N + a}{(1 + z)(1 + w)} - \sum_{i=1}^N \frac{1}{(z - a_i)(w - a_i)} \right) \end{aligned}$$

which which from (3.9) is equivalent to the first equation of the proposition.

Second, we apply Cauchy's integral theorem to Eq. (4.2) and obtain

$$\begin{aligned} \mathcal{K}_{\mathbf{m}}^{\alpha}(x, y) &= \left(\frac{x}{y} \right)^{\alpha/2} e^{(y-x)/2} \sum_{i=1}^D \int_{\mathcal{C}_{\{b_i\}}^{\{-1\}}} \frac{dz}{2\pi i} \frac{e^{-xz}(1 + z)^{|\mathbf{m}|+\alpha}}{\prod_{i=1}^D (z - b_i)^{m_i}} \\ &\times \int_{\mathcal{C}_{\{-1\}}} \frac{dw}{2\pi i} \frac{e^{yw}}{(1 + w)^{|\mathbf{m}|+\alpha}} \prod_{i=1}^D (w - b_i)^{m_i} \frac{1}{w - z} \end{aligned}$$

We can choose the contours $\mathcal{C}_{\mathbf{b}_i}^{\{-1\}}$ and $\mathcal{C}_{\{-1\}}$ in a manner that guarantees $|z - b_i|/|w - b_i| < 1$ and $|1 + w|/|1 + z| < 1$. In such a situation, we have

$$\frac{1}{w - z} = \sum_{n=0}^{\infty} \frac{(z - b_i)^n (1 + w)^n}{(w - b_i)^{n+1} (1 + z)^{n+1}},$$

and consequently,

$$\begin{aligned} \mathcal{K}_{\mathbf{m}}^{\alpha}(x, y) &= \left(\frac{x}{y}\right)^{\alpha/2} e^{(y-x)/2} \sum_{i=1}^D \sum_{j=0}^{\infty} \int_{\mathcal{C}_{\{b_i\}}^{\{-1\}}} \frac{dz}{2\pi i} \frac{e^{-xz}(1+z)^{|\mathbf{m}|+\alpha-j-1}}{(z-b_i)^{m_i-j} \prod_{k \neq i} (z-b_k)^{m_k}} \\ &\quad \times \int_{\mathcal{C}_{\{-1\}}} \frac{dw}{2\pi i} \frac{e^{yw}}{(1+w)^{|\mathbf{m}|+\alpha-j}} (w-b_i)^{m_i-j-1} \prod_{j \neq i} (w-b_k)^{m_k} \end{aligned}$$

The integral in z around b_i has no pole when $j \geq m_i$ and so the sum over j can be truncated. We thus can write

$$\begin{aligned} \mathcal{K}_{\mathbf{m}}^{\alpha}(x, y) &= (xy)^{\alpha/2} e^{-(x+y)/2} \sum_{i=1}^D \sum_{j=0}^{m_i-1} \int_{\mathcal{C}_{\{b_i\}}} \frac{dz}{2\pi i} \frac{e^{-xz}(1+z)^{|\mathbf{m}|+\alpha-j-1}}{(z-b_i)^{m_i-j} \prod_{k \neq i} (z-b_k)^{m_k}} \\ &\quad \times y^{-\alpha} e^{-y} \int_{\mathcal{C}_{\{-1\}}} \frac{dw}{2\pi i} \frac{e^{yw}}{(1+w)^{|\mathbf{m}|+\alpha-j}} (w-b_i)^{m_i-j-1} \prod_{j \neq i} (w-b_k)^{m_k}. \end{aligned}$$

Simple manipulations and recalling (3.9) complete the proof. \square

We now turn our attention to the correlation functions. It is known from [4] that they are given by (4.1) with correlation kernel (4.2). To give further insight into this result, we will re-establish this fact using a formalism analogous to that revised in §2 for the null case. In particular we want to prove Eq. (2.6) remains valid in the non-null case. According to the discussion of §1 the perturbed Laguerre p.d.f. is given by

$$P_N^{\alpha}(x_1, \dots, x_N) = \frac{1}{Z_N^{\alpha}} \prod_{i=1}^N x_i^{\alpha} e^{-x_i} \prod_{1 \leq i < j \leq N} \frac{x_j - x_i}{a_j - a_i} \det [e^{-a_i x_j}]_{i,j=1}^N, \quad (4.5)$$

where it is supposed that no eigenvalues in \mathbf{a} coincide. When $a_i = a_j$ for some $i \neq j$, the appropriate p.d.f. is obtained by applying L'Hospital's rule on $P_N^{\alpha}(x)$. Further, we read off from (1.9) that

$$Z_N^{\alpha} = (-1)^{N(N-1)/2} N! \prod_{i=1}^N (\alpha + i - 1)! \prod_{i=1}^N (1 + a_i)^{-N-\alpha}. \quad (4.6)$$

Lemma 5. *We have $\det [\mathcal{K}_{\mathbf{m}}^{\alpha}(x_i, x_j)]_{i,j=1}^{|\mathbf{m}|} = |\mathbf{m}|! P_{|\mathbf{m}|}^{\alpha}(x_1, \dots, x_{|\mathbf{m}|})$.*

Proof. Let us suppose $a_1 < \dots < a_N$ where $N = |\mathbf{m}|$; i.e., $\mathbf{m} = (1, \dots, 1)$. Then Cauchy's theorem allows us to expand Eq. (4.2),

$$\begin{aligned} \mathcal{K}_{\mathbf{m}}^{\alpha}(x, y) &= \left(\frac{x}{y}\right)^{\alpha/2} e^{(y-x)/2} \sum_{i=1}^N \int_{\mathcal{C}_{\{-1\}}} \frac{dw}{2\pi i} e^{-xa_i+yw} \left(\frac{1+a_i}{1+w}\right)^{N+\alpha} \prod_{\substack{j=1 \\ j \neq i}}^N \left(\frac{w-a_j}{a_i-a_j}\right) \\ &= \left(\frac{x}{y}\right)^{\alpha/2} e^{-(x+y)/2} \sum_{i=1}^N \frac{(1+a_i)^{|\mathbf{m}|+\alpha} e^{-xa_i}}{\prod_{\substack{k=1 \\ k \neq i}}^N (a_i-a_k)} \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi i} \frac{e^{yw}}{w^{N+\alpha}} \prod_{\substack{j=1 \\ j \neq i}}^N (w-1-a_j). \end{aligned}$$

But we have

$$\int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi i} \frac{e^{yw}}{w^n} = \frac{y^{n-1}}{(n-1)!},$$

$$\prod_{\substack{j=1 \\ j \neq i}}^N (w-1-a_j) = \sum_{j=1}^N (-1)^{N-j} e_{N-j}^{(i)}(\mathbf{a}') w^{j-1},$$

where $\mathbf{a}' = \mathbf{1} + \mathbf{a} = (1+a_1, \dots, 1+a_N)$ and where $e_n^{(i)}(\mathbf{x})$ stands for the elementary symmetric function of degree n that does not contain the variable x_i [26, §I.3]. Combining the last equations we get

$$\mathcal{K}_{\mathbf{m}}^\alpha(x, y) = (xy)^{\alpha/2} e^{-(x+y)/2} \sum_{i,j=1}^N e^{-a_i x} C_{i,j} y^{j-1} \quad (4.7)$$

where

$$C_{i,j} = \frac{(-1)^{j-1} (1+a_i)^{N+\alpha}}{(\alpha+j-1)!} e_{j-1}^{(i)}(\mathbf{a}') \prod_{\substack{k=1 \\ k \neq i}}^N (a_i - a_k)^{-1}. \quad (4.8)$$

We finally use

$$\prod_{i < j} (a_j - a_i) = \prod_{i < j} (a'_j - a'_i) = \det [(a'_i)^{j-1}]_{i,j} = \det [e_{j-1}^{(i)}(\mathbf{a}')]_{i,j}$$

and the general formula

$$\det \left[\sum_{k,l} c_{k,l} f_k(x_i) g_l(x_j) \right]_{i,j} = \det [f_i(x_j)]_{i,j} \det [c_{i,j}]_{i,j} \det [g_i(x_j)]_{i,j}$$

to obtain the sought formula. \square

Lemma 6. $\int_0^\infty dx \mathcal{K}_{\mathbf{m}}^\alpha(x, x) = N$ and $\int_0^\infty dy \mathcal{K}_{\mathbf{m}}^\alpha(x, y) \mathcal{K}_{\mathbf{m}}^\alpha(y, z) = \mathcal{K}_{\mathbf{m}}^\alpha(x, z)$.

Proof. From Eqs (4.7) and (4.7), we can write

$$\int_0^\infty dx \mathcal{K}_{\mathbf{m}}^\alpha(x, x) = \sum_{i,j=1}^N C_{i,j} \int_0^\infty dx e^{-x-a_i x} x^{\alpha+j-1} = \sum_{i,j=1}^N C_{i,j} \frac{(\alpha+j-1)!}{(1+a_i)^{\alpha+j}},$$

which can be shown to be equal to N . Similarly

$$\begin{aligned} \int_0^\infty dy \mathcal{K}_{\mathbf{m}}^\alpha(x, y) \mathcal{K}_{\mathbf{m}}^\alpha(y, z) &= (xz)^{\alpha/2} e^{-(x+z)/2} \sum_{i,j,k,l} e^{-a_i x} C_{i,j} C_{k,l} z^{l-1} \int_0^\infty dy e^{-y-a_k y} y^{\alpha+j-1} \\ &= (xz)^{\alpha/2} e^{-(x+z)/2} \sum_{i,j,k,l} e^{-a_i x} C_{i,j} C_{k,l} z^{l-1} \frac{(\alpha+j-1)!}{(1+a_k)^{\alpha+j}} \end{aligned}$$

Direct manipulations then give

$$\sum_{j,k} C_{i,j} C_{k,l} \frac{(\alpha+j-1)!}{(1+a_k)^{\alpha+j}} = C_{i,l}$$

and the proof is completed. \square

Proposition 7. *Let $\rho_n^\alpha(x_1, \dots, x_n)$ be the n point correlation function in the multiple Laguerre ensemble; that is,*

$$\rho_n^\alpha(x_1, \dots, x_n) := \frac{|\mathbf{m}|!}{(|\mathbf{m}| - n)!} \int_0^\infty dx_{n+1} \cdots \int_0^\infty dx_{|\mathbf{m}|} P_{|\mathbf{m}|}^\alpha(x_1, \dots, x_{|\mathbf{m}|}).$$

Then

$$\rho_n^\alpha(x_1, \dots, x_n) = \det [\mathcal{K}_{\mathbf{m}}^\alpha(x_i, x_j)]_{i,j=1}^n.$$

Proof. On the one hand, we know from Lemma 5 that

$$\rho_n^\alpha(x_1, \dots, x_n) := \frac{1}{(|\mathbf{m}| - n)!} \int_0^\infty dx_{n+1} \cdots \int_0^\infty dx_{|\mathbf{m}|} \det [\mathcal{K}_{\mathbf{m}}^\alpha(x_i, x_j)]_{i,j=1}^{|\mathbf{m}|}.$$

On the other hand, Lemma 6 and the Laplace expansion of the determinant imply that

$$\int_0^\infty dx_k \det [\mathcal{K}_{\mathbf{m}}^\alpha(x_i, x_j)]_{i,j=1}^k = (|\mathbf{m}| + 1 - k) \det [\mathcal{K}_{\mathbf{m}}^\alpha(x_i, x_j)]_{i,j=1}^{k-1}.$$

Thus, starting with $k = |\mathbf{m}|$ and applying $|\mathbf{m}| - n - 1$ times the last equation on $\det [\mathcal{K}_{\mathbf{m}}^\alpha(x_i, x_j)]_{i,j=1}^{|\mathbf{m}|}$ give the formula we wanted to prove. \square

4.2. Multiple Hermite kernel. The appropriate generalized Hermite kernel has been given in double contour integral form by Zinn-Justin [33]. With our notation and normalization, it reads

$$\mathcal{K}_{\mathbf{m}}(x, y) = e^{y^2/2 - x^2/2} \int_{\mathcal{C}_b} \frac{dz}{2\pi i} \int_{-\infty}^{\infty} \frac{dw}{2\pi i} \frac{e^{w^2/4 - z^2/4} e^{-xz + yw}}{w - z} \prod_{i=1}^D \left(\frac{w - b_i}{z - b_i} \right)^{m_i}, \quad (4.9)$$

where it is understood that z never crosses the w 's path.

The following proposition will considerably simplify our analysis of the multiple Hermite kernel. It is proved by following the method exposed in Proposition 2.

Proposition 8. *Let X be a real number such that $|\sqrt{2\alpha}X| < \alpha$. Then, as $\alpha \rightarrow \infty$,*

$$\sqrt{2\alpha} \mathcal{K}_{\mathbf{m}}^\alpha(\alpha + \sqrt{2\alpha}X, \alpha + \sqrt{2\alpha}Y) \Big|_{b_i \mapsto b_i/\sqrt{2\alpha}} = \mathcal{K}_{\mathbf{m}}(X, Y) + \mathcal{O}\left(\frac{1}{\sqrt{\alpha}}\right).$$

Corollary 9. *We have*

$$\mathcal{K}_{\mathbf{m}}(x, y) = \frac{-1}{2^{|\mathbf{m}|} \sqrt{\pi} (|\mathbf{m}| - 1)!} \frac{e^{-(x^2 + y^2)/2}}{x - y} \left(\tilde{\mathcal{H}}_{\mathbf{m}}(x) \mathcal{H}_{\mathbf{m}}(y) - \sum_{i=1}^D \frac{m_i}{|\mathbf{m}|} \tilde{\mathcal{H}}_{\mathbf{m} + \mathbf{e}_i}(x) \mathcal{H}_{\mathbf{m} - \mathbf{e}_i}(y) \right)$$

and

$$\mathcal{K}_{\mathbf{m}}(x, y) = \frac{e^{-(x^2 + y^2)/2}}{2^{|\mathbf{m}|} \sqrt{\pi}} \sum_{i=1}^D \sum_{j=1}^{m_i} \frac{2^j e^{-b_i x}}{(|\mathbf{m}| - j)!} \tilde{\mathcal{H}}_{\mathbf{m} - (j-1)\mathbf{e}_i}(x) \mathcal{H}_{\mathbf{m} + j\mathbf{e}_i}(y).$$

Proof. We first make the following change of variable in Proposition 4: $x \mapsto \alpha + \sqrt{2\alpha}x$, $\alpha + \sqrt{2\alpha}y$, and $b_i \mapsto b_i/\sqrt{2\alpha}$. Then we use the asymptotics of the multiple Laguerre functions given in Proposition 2 to write the scaled perturbed Laguerre kernel in terms of multiple Hermite functions. The proof ends with the comparison of the latter expression and the scaled kernel of Proposition 8. \square

In a similar way we can prove that the correlation functions in the perturbed Hermite ensemble are determinants of the multiple Hermite kernel. We recall that this ensemble is characterized by the p.d.f.

$$P_N(x_1, \dots, x_N) = \frac{1}{Z_N} \prod_{i=1}^N e^{-x_i^2} \prod_{1 \leq i < j \leq N} \frac{x_j - x_i}{a_j - a_i} \det [e^{-a_i x_j}]_{i,j=1}^N. \quad (4.10)$$

The normalization constant Z_N is read off from (1.7),

$$Z_N = \frac{(-1)^{N(N-1)/2} (2\pi)^{N/2} N!}{2^{N^2/2}} \prod_{i=1}^N e^{a_i^2/4}.$$

Corollary 10. *Let*

$$\rho_n(x_1, \dots, x_n) := \frac{|\mathbf{m}|!}{(|\mathbf{m}| - n)!} \int_{-\infty}^{\infty} dx_{n+1} \cdots \int_{-\infty}^{\infty} dx_{|\mathbf{m}|} P_{|\mathbf{m}|}(x_1, \dots, x_{|\mathbf{m}|}).$$

Then

$$\rho_n(x_1, \dots, x_n) = \det [\mathcal{K}_{\mathbf{m}}(x_i, x_j)]_{i,j=1}^n.$$

Proof. The expressions of the Laguerre and Hermite p.d.f. (respectively given in Eqs (4.5) and (4.10)) together with the Stirling approximation allow us to write

$$\lim_{\alpha \rightarrow \infty} \left\{ (2\alpha)^{N/2} P_N^\alpha(x_1, \dots, x_N) \Big|_{\substack{x_i \mapsto \alpha + \sqrt{2\alpha} x_i \\ b_i \mapsto b_i / \sqrt{2\alpha}}} \right\} = P_N(x_1, \dots, x_N).$$

Hence the corollary is an immediate consequence of Propositions 7 and 8. \square

5. QUASI-NULL CASE

In this section, we suppose that all but a fixed number of eigenvalues of the matrix \mathbf{A} are zero. Precisely,

$$\begin{aligned} \mathbf{a} &= (a_1, \dots, a_r, 0, \dots, 0), \\ \mathbf{b} &= (b_1, \dots, b_d, 0), \\ \mathbf{m} &= (m_1, \dots, m_d, N - r). \end{aligned} \quad (5.1)$$

Here $d = D - 1$ gives the number of distinct parameters in \mathbf{a} that are not equal to zero. To make contact with the results of [4] and [29] reviewed in §1, we seek limiting expressions for the kernels, and thus for the correlations, when the size N of the random matrices becomes infinite.

5.1. Perturbed Laguerre ensemble. The kernel is now given by

$$\mathcal{K}_{\mathbf{m}}^\alpha(x, y) = \sqrt{\frac{w_\alpha(x)}{w_\alpha(y)}} \int_{\mathcal{C}_{\mathbf{a}}^{\{-1\}}} \frac{dz}{2\pi i} \int_{\mathcal{C}_{\{-1\}}} \frac{dw}{2\pi i} \frac{e^{-xz+yw}}{w-z} \left(\frac{1+z}{1+w} \right)^{N+\alpha} \left(\frac{w}{z} \right)^{N-r} \prod_{i=1}^r \frac{w - a_i}{z - a_i}, \quad (5.2)$$

where $N = |\mathbf{m}|$ and $w_\alpha(x) = x^\alpha e^{-x}$, as usual. It is convenient to introduce

$$\bar{\mathcal{K}}_{\mathbf{m}}^\alpha(x, y) := \sqrt{\frac{w_\alpha(y)}{w_\alpha(x)}} \mathcal{K}_{\mathbf{m}}^\alpha(x, y). \quad (5.3)$$

This gauge transformation does not affect the correlation functions because

$$\det \left[\frac{f(x_i)}{f(x_j)} g(x_i, y_j) \right]_{i,j=1}^N = \det [g(x_i, y_j)]_{i,j=1}^N$$

for all $g(x_i, y_j)$ and $f(x) \neq 0$.

A striking feature of the quasi-null soft edge scaled correlation (1.12) is that the correlation kernel consists of the null case Airy kernel (1.14) plus r perturbation terms. As our key result we now show that at the level of finite N , the multiple Laguerre kernel can be decomposed as a (null case) Laguerre kernel plus r perturbation terms. The latter are written in terms of incomplete multiple Laguerre functions of type I and II,

$$\tilde{\Lambda}^{(i)}(x) := \int_{\mathcal{C}_{\{0, a_1, \dots, a_i\}}} \frac{dz}{2\pi i} \frac{e^{-xz}(1+z)^{N+\alpha}}{z^{N-r} \prod_{k=1}^i (z - a_k)} \quad (5.4)$$

and

$$\Lambda^{(i)}(x) := \int_{\mathcal{C}_{\{-1\}}} \frac{dw}{2\pi i} \frac{e^{xw} w^{N-r} \prod_{k=1}^{i-1} (w - a_k)}{(1+w)^{N+\alpha}} \quad (5.5)$$

respectively. Comparison with (3.9) shows that

$$\tilde{\Lambda}^{(i)}(x) = \tilde{\mathcal{L}}_{\mathbf{m}'}^{\alpha+r+1-i}(x|\mathbf{b}')$$

when

$$\mathbf{b}' = \{a_1, \dots, a_i, 0\} \quad \text{and} \quad \mathbf{m}' = \{\overbrace{1, \dots, 1}^i, N-r\}. \quad (5.7)$$

Similarly, we see from (3.10) that

$$\Lambda^{(i+1)}(x) = \frac{(N+i-r)!}{(N+\alpha-1)!} x^{\alpha+r-i-1} e^{-x} \mathcal{L}_{\mathbf{m}'}^{\alpha+r-i-1}(x|\mathbf{b}'). \quad (5.8)$$

As $\tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(x)$ and $\mathcal{L}_{\mathbf{m}}^{\alpha}(x)$ can be interpreted as anti-derivatives and derivations of Laguerre polynomials according to (3.18) and (3.19), it follows from (5.6) and (5.8) that $\tilde{\Lambda}^{(i)}(x)$ and $\Lambda^{(i)}(x)$ are also simply related to Laguerre polynomials. Explicitly,

$$\mathcal{D}_{-a_1, x} \cdots \mathcal{D}_{-a_i, x} [\tilde{\Lambda}^{(i)}(x)] = L_{N-r-1}^{\alpha+r+1}(x)$$

and

$$\Lambda^{(i)}(x) = (-1)^{\alpha+r+i} e^{-x} \mathcal{D}_{a_1, x} \cdots \mathcal{D}_{a_{i-1}, x} [L_{N-\alpha+1}^{-\alpha-r+1}(x)].$$

where in deriving the latter use has also been made of (3.12).

Proposition 11. *Let $\bar{K}_N^{\alpha}(x, y)$ be the gauged transformed Laguerre kernel; that is,*

$$\bar{K}_N^{\alpha}(x, y) = \lim_{\mathbf{a} \rightarrow \mathbf{0}} \bar{K}_{\mathbf{m}}^{\alpha}(x, y) = \int_{\mathcal{C}_{\{0\}}^{\{-1\}}} \frac{dz}{2\pi i} \int_{\mathcal{C}_{\{-1\}}} \frac{dw}{2\pi i} \frac{e^{-xz+yw}}{w-z} \left(\frac{1+z}{1+w} \right)^{N+\alpha} \left(\frac{w}{z} \right)^N$$

for $N = |\mathbf{m}|$ and $\mathcal{C}_{\{0\}}^{\{-1\}} \cap \mathcal{C}_{\{-1\}} = \emptyset$. Then

$$\bar{K}_{\mathbf{m}}^{\alpha}(x, y) = \bar{K}_{N-r}^{\alpha+r}(x, y) + \sum_{i=1}^r \tilde{\Lambda}^{(i)}(x) \Lambda^{(i)}(y). \quad (5.9)$$

Proof. We first consider the relation

$$\frac{1}{w-z} \prod_{i=1}^r \frac{w-a_i}{z-a_i} = \frac{1}{w-z} + \sum_{i=1}^r \frac{\prod_{k=1}^{i-1} w-a_k}{\prod_{k=1}^i z-a_k}. \quad (5.10)$$

For $r = 1$, this reads

$$\frac{1}{w-z} \frac{w-a_1}{z-a_1} = \frac{1}{w-z} + \frac{1}{z-a_1},$$

which is trivially true. For $r > 1$, Eq. (5.10) is proved by induction; explicitly,

$$\begin{aligned} \frac{1}{w-z} \prod_{i=1}^{r+1} \frac{w-a_i}{z-a_i} &= \frac{1}{w-z} \left(\prod_{i=1}^r \frac{w-a_i}{z-a_i} \right) \frac{z-a_{r+1}+w-z}{z-a_{r+1}} \\ &= \frac{1}{w-z} \prod_{i=1}^r \frac{w-a_i}{z-a_i} + \left(\prod_{i=1}^r \frac{w-a_i}{z-a_i} \right) \frac{1}{z-a_{r+1}} \\ &= \frac{1}{w-z} + \sum_{i=1}^r \frac{\prod_{k=1}^{i-1} w-a_k}{\prod_{k=1}^i z-a_k} + \frac{\prod_{k=1}^r w-a_k}{\prod_{k=1}^{r+1} z-a_k} \end{aligned}$$

as expected.

We then substitute Eq. (5.10) in the quasi-null kernel, given in Eq. (5.2) (see also Eq. (5.3)). This gives

$$\begin{aligned} \bar{\mathcal{K}}_{\mathbf{m}}^{\alpha}(x, y) &= \int_{\mathcal{C}_{\{0\}}^{\{-1\}}} \frac{dz}{2\pi i} \int_{\mathcal{C}_{\{-1\}}} \frac{dw}{2\pi i} \frac{e^{-xz+yw}}{w-z} \left(\frac{1+z}{1+w} \right)^{N+\alpha} \left(\frac{w}{z} \right)^{N-r} \\ &\quad + \sum_{i=1}^r \int_{\mathcal{C}_{\{a_1, \dots, a_i, 0\}}} \frac{dz}{2\pi i} \frac{e^{-xz}(1+z)^{N+\alpha}}{z^{N-r} \prod_{k=1}^i (z-a_k)} \int_{\mathcal{C}_{\{-1\}}} \frac{dw}{2\pi i} \frac{e^{yw} w^{N-r} \prod_{k=1}^{i-1} w-a_k}{(1+w)^{N+\alpha}}. \end{aligned}$$

Comparing the latter equation with Eqs (5.4), (5.5) finally completes the proof. \square

5.2. Perturbed Hermite kernel. When the set \mathbf{a} of eigenvalues is quasi-null, the multiple Hermite kernel becomes

$$\mathcal{K}_{\mathbf{m}}(x, y) = e^{y^2/2-x^2/2} \int_{\mathcal{C}_{\mathbf{a}}} \frac{dz}{2\pi i} \int_{-\infty}^{\infty} \frac{dw}{2\pi i} \frac{e^{w^2/4-z^2/4} e^{-xz+yw}}{w-z} \left(\frac{w}{z} \right)^{N-r} \prod_{i=1}^r \left(\frac{w-a_i}{z-a_i} \right). \quad (5.11)$$

Note that the w 's contour integral does not pass through the origin. As in the perturbed Laguerre ensemble, we introduce incomplete multiple Hermite functions,

$$\tilde{\Gamma}^{(i)}(x) := e^{-x^2/2} \int_{\mathcal{C}_{\{0, a_1, \dots, a_i\}}} \frac{dz}{2\pi i} \frac{e^{-xz-z^2/4}}{z^{N-r} \prod_{k=1}^i (z-a_k)} \quad (5.12)$$

and

$$\Gamma^{(i)}(x) := e^{x^2/2} \int_{-\infty}^{\infty} \frac{dw}{2\pi i} e^{xw+w^2/4} w^{N-r} \prod_{k=1}^{i-1} (w-a_k). \quad (5.13)$$

Comparison with (3.20) and (3.21) shows

$$\begin{aligned}\tilde{\Gamma}^{(i)}(x) &= \frac{e^{-x^2/2}}{(-2)^{N+i-r-1}(N+i-r-1)!} \tilde{\mathcal{H}}_{\mathbf{m}'}(x|\mathbf{b}'), \\ \Gamma^{(i+1)}(x) &= \frac{(-1)^{N+i-r}}{\sqrt{\pi}} e^{-x^2/2} \mathcal{H}_{\mathbf{m}'}(x|\mathbf{b}'),\end{aligned}\tag{5.14}$$

where \mathbf{b}' and \mathbf{m}' are the restricted sets of parameters given in (5.7). Consequently, we read off from (3.23), (3.24) that these special multiple Hermite polynomials are related to the classical Hermite polynomials by

$$H_{N-r-1} = (-2)^{N-r-1} (N-r-1)! \mathcal{D}_{-a_1, x} \cdots \mathcal{D}_{-a_i, x} \left[e^{x^2/2} \tilde{\Gamma}^{(i)}(x) \right]$$

and

$$\Gamma^{(i)}(x) = \frac{(-1)^{N+r+i+1}}{\sqrt{\pi}} e^{x^2/2} \mathcal{D}_{a_1, x} \cdots \mathcal{D}_{a_{i-1}, x} \left[e^{-x^2/2} H_{N-r}(x) \right].$$

The algebraic property given in Eq. (5.10) can be used to prove the following proposition.

Proposition 12. *Let $K_N(x, y)$ be the usual Hermite kernel; that is,*

$$K_N(x, y) = \lim_{\mathbf{a} \rightarrow \mathbf{0}} \mathcal{K}_{\mathbf{m}}(x, y) = e^{y^2/2 - x^2/2} \int_{\mathcal{C}_{\{0\}}} \frac{dz}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} \frac{e^{w^2/4 - z^2/4} e^{-xz + yw}}{w - z} \left(\frac{w}{z} \right)^N.$$

Then

$$\mathcal{K}_{\mathbf{m}}(x, y) = K_{N-r}(x, y) + \sum_{i=1}^r \tilde{\Gamma}^{(i)}(x) \Gamma^{(i)}(y).$$

Before considering the large N limit of the kernels, let us link the perturbed Laguerre and Hermite ensembles by relating the incomplete multiple Laguerre and Hermite polynomials.

Proposition 13. *Set X a real number satisfying $|\sqrt{2\alpha}X| < \alpha$. Then, as $\alpha \rightarrow \infty$,*

$$(2\alpha)^{(1+r-i-N)/2} \tilde{\Lambda}^{(i)}(\alpha + \sqrt{2\alpha}X) \Big|_{a_i \mapsto a_i/\sqrt{2\alpha}} = e^{X^2/2} \tilde{\Gamma}^{(i)}(X) + \mathcal{O}\left(\frac{1}{\sqrt{\alpha}}\right)$$

and

$$(2\alpha)^{(N+i-r)/2} \Lambda^{(i)}(\alpha + \sqrt{2\alpha}Y) \Big|_{a_i \mapsto a_i/\sqrt{2\alpha}} = e^{-Y^2/2} \Gamma^{(i)}(Y) + \mathcal{O}\left(\frac{1}{\sqrt{\alpha}}\right).$$

Proof. We essentially follow the steps given in the proof of Proposition 2. We skip the detail. \square

The latter result obviously complies with Proposition 8 and the asymptotics

$$\sqrt{\frac{w_\alpha(\alpha + \sqrt{2\alpha}X)}{w_\alpha(\alpha + \sqrt{2\alpha}Y)}} = e^{(Y^2 - X^2)/2} + \mathcal{O}\left(\frac{1}{\sqrt{\alpha}}\right).$$

6. KERNELS AT THE EDGES OF THE SPECTRUM

6.1. Hard edge. We know from (1.14) that the null-case hard edge kernel can be written in terms of Bessel functions $J_\alpha(x)$. The latter satisfy

$$J_\alpha(x) = \int_{\mathcal{C}_{\{0\}}} \frac{dz}{2\pi i} \frac{e^{x(z-z^{-1})/2}}{z^{\alpha+1}}, \quad J_{-\alpha}(x) = (-1)^\alpha J_\alpha(x), \quad \alpha \in \mathbb{Z}.\tag{6.1}$$

The form (5.9) of the perturbed Laguerre kernel suggests that its hard edge scaling can be expressed in terms of the Bessel like functions (1.20), (1.21). Use of (3.15) and (6.1) shows these functions are simply related to the Bessel function by

$$J_{\alpha+r+1}(\sqrt{x}) = (4x)^{(\alpha+r+1)/2} \mathcal{D}_{-h_1, x} \cdots \mathcal{D}_{-h_i, x} \left[\tilde{J}^{(i)}(x) \right]$$

and

$$J^{(i)}(x) = (-1)^{i-1} \mathcal{D}_{h_1, x} \cdots \mathcal{D}_{h_{i-1}, x} \left[x^{(\alpha+r-1)/2} J_{\alpha+r-1}(\sqrt{x}) \right].$$

The functions $\tilde{J}^{(i)}$ and $J^{(i)}$ are in fact incomplete versions of the multiple Bessel functions of type I and II, which we respectively define as

$$\tilde{\mathcal{J}}_{\mathbf{m}}^{\alpha}(x) = \tilde{\mathcal{J}}_{\mathbf{m}}^{\alpha}(x|\boldsymbol{\nu}) := (2x)^{\alpha} \int_{\mathcal{C}_{\{0\} \cup \boldsymbol{\nu}}} \frac{dz}{2\pi i} \frac{z^{|\mathbf{m}|+\alpha-1} e^{-x^2 z + 1/(4z)}}{\prod_{i=1}^D (z - \nu_i)^{m_i}} \quad (6.2)$$

and

$$\mathcal{J}_{\mathbf{m}}^{\alpha}(x) = \mathcal{J}_{\mathbf{m}}^{\alpha}(x|\boldsymbol{\nu}) := (2x)^{-\alpha} \int_{\mathcal{C}_{\{0\}}} \frac{dw}{2\pi i} \frac{e^{x^2 w - 1/(4w)} \prod_{i=1}^D (w - \nu_i)^{m_i}}{w^{|\mathbf{m}|+\alpha+1}}. \quad (6.3)$$

Recall that D denotes, as in Eqs. (3.1) and (3.2), the number of distinct parameters in $\mathbf{h} = \boldsymbol{\nu}^{\mathbf{m}}$. The link relation between the incomplete and the complete functions is obvious,

$$\tilde{\mathcal{J}}_{\{1^i\}}^{\alpha+r+1-i}(\sqrt{x}) = (2\sqrt{x})^{\alpha+r+1-i} \tilde{J}^{(i)}(x) \quad \text{and} \quad \mathcal{J}_{\{1^{i-1}\}}^{\alpha+r-i}(\sqrt{x}) = (2\sqrt{x})^{-\alpha-r+i} J^{(i)}(x), \quad (6.4)$$

where $\{1^k\}$ denotes a composition of length k in which all entries being equal to unity.

Lemma 14. *Suppose that Eq. (5.1) holds. Let $b_i = 4N\nu_i$ for $i = 1, \dots, d$ and $\boldsymbol{\mu} = (m_1, \dots, m_d)$. Then, as $N \rightarrow \infty$,*

$$\begin{aligned} (4N)^{-\alpha} \tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(X/4N) &= (2\sqrt{X})^{-\alpha} \tilde{\mathcal{J}}_{\boldsymbol{\mu}}^{\alpha}(\sqrt{X}) + O(1/N), \\ (4N)^{-\alpha} \mathcal{L}_{\mathbf{m}}^{\alpha}(X/4N) &= (2\sqrt{X})^{-\alpha} \mathcal{J}_{\boldsymbol{\mu}}^{\alpha}(\sqrt{X}) + O(1/N). \end{aligned}$$

Similarly, for $a_i = 4Nh_i$ ($i = 1, \dots, r$), we have

$$\begin{aligned} (4N)^{i-\alpha-r-1} \tilde{\Lambda}^{(i)}(X/4N) &= \tilde{J}^{(i)}(X) + O(1/N), \\ (4N)^{\alpha+r-i} \Lambda^{(i)}(X/4N) &= J^{(i)}(X) + O(1/N). \end{aligned}$$

Proof. From the definition of $\tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}$, given in Eq (3.9), we have

$$\tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(X/4N) = \int_{\mathcal{C}_{\{0, b_1, \dots, b_d\}}^{\{-1\}}} \frac{dz}{2\pi i} \frac{e^{-Xz/4N} (1+z)^{N+\alpha-1}}{z^{N-r} \prod_{k=1}^d (z - b_k)^{m_k}}.$$

Then, changing z for $4Nw$ and b_i for $4N\nu_i$, we get

$$\tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(X/4N) = (4N)^{\alpha} \int_{\mathcal{C}_{\{0, h_1, \dots, h_d\}}} \frac{dw}{2\pi i} \frac{e^{-Xw} w^{\alpha+r-1} (1+1/4Nw)^{N+\alpha}}{\prod_{k=1}^d (w - \nu_k)^{m_k}}.$$

But for $N \rightarrow \infty$,

$$(1 + 1/4Nw)^{N+\alpha} = (1 + 1/4Nw)^N + O(1/N) = e^{1/4w} + O(1/N).$$

This immediately implies that

$$\tilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(X/4N) = (4N)^{\alpha} \int_{\mathcal{C}_{\{0, h_1, \dots, h_d\}}} \frac{dw}{2\pi i} \frac{e^{-Xw+1/4w} w^{\alpha+r-1}}{\prod_{k=1}^d (w - \nu_k)^{m_k}} + O(N^{\alpha-1}).$$

Comparing with Eq. (6.3) gives the desired result. The proof for the other functions is similar. \square

Theorem 15. *Suppose that Eq. (5.1) holds. Set $a_i = 4Nh_i$ for $i = 1, \dots, q$ and $a_i = (4N)^2 h_i$ for $i = q+1, \dots, r$. Following the notation of §3, let $\mathbf{h} := \{h_1, \dots, h_q\} = \{\nu_i^{\mu_i}, \dots, \nu_p^{\mu_p}\} =: \boldsymbol{\nu}^{\boldsymbol{\mu}}$ with $q = |\boldsymbol{\mu}|$. Then, as $N \rightarrow \infty$,*

$$\frac{1}{4N} \bar{\mathcal{K}}_{\mathbf{m}}^{\alpha} \left(\frac{X}{4N}, \frac{Y}{4N} \right) = \mathcal{K}_{\boldsymbol{\mu}}^{\text{hard}}(X, Y) + O\left(\frac{1}{N}\right),$$

where

$$\mathcal{K}_{\boldsymbol{\mu}}^{\text{hard}}(X, Y) = \left(\frac{Y}{X} \right)^{(\alpha+r)/2} K_{\alpha+r}^{\text{hard}}(X, Y) + \sum_{i=1}^q \tilde{J}^{(i)}(X) J^{(i)}(Y) \quad (6.5)$$

$$= \int_{\mathcal{C}_{\{0\} \cup \boldsymbol{\nu}}} \frac{dz}{2\pi i} \int_{\mathcal{C}_{\{0\}}^{\{z\}}} \frac{dw}{2\pi i} \frac{e^{-Xz+1/(4z)} e^{Yw-1/(4w)}}{w-z} \left(\frac{z}{w} \right)^{\alpha+r} \prod_{i=1}^p \left(\frac{w - \nu_i}{z - \nu_i} \right)^{\mu_i} \quad (6.6)$$

$$= \frac{1}{2(X-Y)} \left(\frac{Y}{X} \right)^{\alpha'/2} \left(J(X, Y) - 2 \sum_{i=1}^q \mu_i \tilde{\mathcal{J}}_{\boldsymbol{\mu}+\mathbf{e}_i}^{\alpha'}(\sqrt{X}) \mathcal{J}_{\boldsymbol{\mu}-\mathbf{e}_i}^{\alpha'}(\sqrt{Y}) \right). \quad (6.7)$$

with $\alpha' = \alpha + r - q$ and

$$J(X, Y) = \tilde{\mathcal{J}}_{\boldsymbol{\mu}}^{\alpha'}(\sqrt{X}) \left((\alpha' + q) \mathcal{J}_{\boldsymbol{\mu}}^{\alpha'}(\sqrt{Y}) - \sqrt{Y} \mathcal{J}_{\boldsymbol{\mu}}^{\alpha'+1}(\sqrt{Y}) \right) \\ + \mathcal{J}_{\boldsymbol{\mu}}^{\alpha'}(\sqrt{Y}) \left((\alpha' + q) \tilde{\mathcal{J}}_{\boldsymbol{\mu}}^{\alpha}(\sqrt{X}) - \sqrt{X} \tilde{\mathcal{J}}_{\boldsymbol{\mu}}^{\alpha'-1}(\sqrt{X}) \right). \quad (6.8)$$

Proof. Let us begin with Eq. (6.5). We know from the previous lemma that

$$\tilde{\Lambda}^{(i)}(X/4N) \Lambda^{(i)}(Y/4N) = (4N) \tilde{J}^{(i)}(X) J^{(i)}(Y) + O(1) \quad \text{when } i = 1, \dots, q.$$

For $i > q$, Lemma 16 and Eqs (1.19)–(1.20) imply

$$\tilde{\Lambda}^{(i+q)}(X/4N) \Lambda^{(i+q)}(Y/4N) = \tilde{J}^{(q)}(X) J^{(q)}(Y) + O\left(\frac{1}{N}\right) \quad \text{when } i = 1, \dots, r - q.$$

Furthermore, we have from Eq. (5.3) and §2 (see also [15, §4])

$$\frac{1}{4N} \bar{K}_N^{\alpha+r} \left(\frac{X}{4N}, \frac{Y}{4N} \right) = \left(\frac{X}{Y} \right)^{(\alpha+r)/2} K_{\alpha+r}^{\text{hard}}(X, Y) + O\left(\frac{1}{N}\right),$$

where the hard edge kernel is defined in Eq. (1.19). We finally obtain Eq. (6.5) by using Proposition 11.

We now turn our attention to Eq. (6.6). We first claim that the hard edge kernel has the double integral representation

$$K_{\alpha}^{\text{hard}}(x, y) = \left(\frac{x}{y} \right)^{\alpha/2} \int_{\mathcal{C}_{\{0\}}} \frac{dz}{2\pi i} \int_{\mathcal{C}_{\{0\}}^{\{z\}}} \frac{dw}{2\pi i} \frac{e^{-xz+1/(4z)} e^{yw-1/(4w)}}{w-z} \left(\frac{z}{w} \right)^{\alpha}.$$

This is proved by effectively multiplying the previous equation by $(x-y)/(x-y)$ and integrating by parts; that is,

$$\begin{aligned} K_\alpha^{\text{hard}}(x, y) &= \frac{1}{x-y} \left(\frac{x}{y}\right)^{\alpha/2} \int_{\mathcal{C}_{\{0\}}} \frac{dz}{2\pi i} \int_{\mathcal{C}_{\{z\}}} \frac{dw}{2\pi i} \frac{e^{1/(4z)-1/(4w)}}{w-z} \left(\frac{z}{w}\right)^\alpha \left(-\frac{\partial}{\partial z} - \frac{\partial}{\partial w}\right) (e^{-xz+yw}), \\ &= \frac{1}{x-y} \left(\frac{x}{y}\right)^{\alpha/2} \int_{\mathcal{C}_{\{0\}}} \frac{dz}{2\pi i} \int_{\mathcal{C}_{\{z\}}} \frac{dw}{2\pi i} e^{1/(4z)-1/(4w)} e^{-xz+yw} \left(\frac{z}{w}\right)^\alpha \left(\frac{\alpha}{zw} - \frac{z+w}{4z^2w^2}\right), \\ &= \frac{1}{2(x-y)} (2\alpha J_\alpha(\sqrt{x}) J_\alpha(\sqrt{y}) - \sqrt{x} J_{\alpha-1}(\sqrt{x}) J_\alpha(\sqrt{y}) - J_\alpha(\sqrt{x}) \sqrt{y} J_{\alpha+1}(\sqrt{y})), \end{aligned}$$

which, by virtue of $\sqrt{x} J'_\alpha(\sqrt{x}) = \sqrt{x} J_{\alpha-1}(\sqrt{x}) - \alpha J_\alpha(\sqrt{x})$ and $\sqrt{x} J_{\alpha+1}(\sqrt{x}) = -\sqrt{x} J_{\alpha-1}(\sqrt{x}) + 2\alpha J_\alpha(\sqrt{x})$, turns out to be equivalent to Eq. (1.19). Then we use the integral representations of the incomplete multiple Bessel functions, which are given in Eqs (1.20)–(1.21), to rewrite Eq. (6.5) as

$$\begin{aligned} \mathcal{K}_\alpha^{\text{hard}}(X, Y) &= \int_{\mathcal{C}_{\{0\} \cup \mathbf{h}}} \frac{dz}{2\pi i} \int_{\mathcal{C}_{\{z\}}} \frac{dw}{2\pi i} e^{-Xz+1/(4z)} e^{Yw-1/(4w)} \\ &\quad \times \left(\frac{z}{w}\right)^{\alpha+r} \left(\frac{1}{w-z} + \sum_{i=1}^q \prod_{k=1}^i \left(\frac{w-h_k}{z-h_k} \right) \frac{1}{w-h_i} \right), \end{aligned}$$

where the Cauchy theorem has been applied for deforming the contours $\mathcal{C}_{\{0\}}$ and $\mathcal{C}_{\{0, h_1, \dots, h_i\}}$ into $\mathcal{C}_{\{0\} \cup \mathbf{h}}$. Use of Eq. (5.10) then finishes the proof of formula Eq. (6.6).

We show Eq. (6.7) with the same method than the one explained in the proof of Proposition 4: we multiply Eq. (6.6) by $(X-Y)^{-1}(X-Y) = -(X-Y)^{-1}e^{Xz-Yw}(\partial_z + \partial_w)e^{-Xz+Yw}$; we integrate by parts; and we compare the result with the definition of the multiple Bessel functions, Eqs (6.2)–(6.3). \square

6.2. Soft edge. Working directly with the double contour form (5.2) of the correlation kernel Baik, Ben Arous and P      [4] have shown that the soft edge scaling (1.10) of perturbed Laguerre kernel can be written in terms of multiple Airy functions according to the results (1.10)–(1.16). From the contour integral representation of the Airy function,

$$\text{Ai}(x) := \int_{\mathcal{A}} \frac{dv}{2\pi i} e^{-xv+v^3/3}, \quad (6.9)$$

and (3.15) we see that the multiple functions (1.14), (1.15) are related to the Airy function by

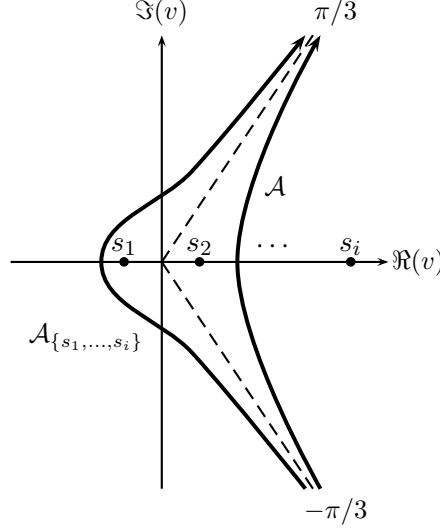
$$\text{Ai}(x) = \mathcal{D}_{-s_1} \cdots \mathcal{D}_{-s_i} \left[\widetilde{\text{Ai}}^{(i)}(x) \right]$$

and

$$\text{Ai}^{(i)}(x) = (-1)^i \mathcal{D}_{s_1} \cdots \mathcal{D}_{s_{i-1}} [\text{Ai}(x)].$$

According to §6.1, the functions $\text{Ai}^{(i)}$ and $\widetilde{\text{Ai}}^{(i)}$ can be completed in order to get the multiple Airy functions of types I and II,

$$\widetilde{\mathcal{A}}_{\mathbf{m}}(x) = \widetilde{\mathcal{A}}_{\mathbf{m}}(x|\boldsymbol{\sigma}) := \int_{\mathcal{A}_{\boldsymbol{\sigma}}} \frac{dv}{2\pi i} e^{-xv+v^3/3} \prod_{k=1}^D (v - \sigma_k)^{-m_k}, \quad (6.10)$$

FIGURE 2. Contours \mathcal{A} and $\mathcal{A}_{\{s_1, \dots, s_i\}}$.

and

$$\mathcal{A}i_{\mathbf{m}}(x) = \mathcal{A}i_{\mathbf{m}}(x|\boldsymbol{\sigma}) := (-1)^{|\mathbf{m}|} \int_{\mathcal{A}} \frac{dv}{2\pi i} e^{-xv+v^3/3} \prod_{k=1}^D (v + \sigma_k)^{m_k}. \quad (6.11)$$

One can easily relate the two sets of multiple Airy functions, follows:

$$\widetilde{\mathcal{A}i}_{\{1^i\}}(x) = \widetilde{\text{Ai}}^{(i)}(x) \quad \text{and} \quad \mathcal{A}i_{\{1^{i-1}\}}(x) = -\text{Ai}^{(i)}(x). \quad (6.12)$$

Lemma 16. Suppose that Eq. (5.1) holds. Let $A = 4N$ and $B = 2(2N)^{1/3}$. Let also $b_i = -1/2 + \sigma_i/B$ for $i = 1, \dots, d$, $\boldsymbol{\mu} = (m_1, \dots, m_d)$ and $|\boldsymbol{\mu}| = r$. Then, as $N \rightarrow \infty$,

$$\begin{aligned} B^{1-r} \widetilde{\mathcal{L}}_{\mathbf{m}}^{\alpha}(A + BX) &= \frac{(-1)^{N+r+1}}{2^{\alpha+r-1}} \widetilde{\mathcal{A}i}_{\boldsymbol{\mu}}(X) + \mathcal{O}\left(\frac{1}{N^{1/3}}\right) \\ B^{1+r} \mathcal{L}_{\mathbf{m}}^{\alpha}(A + BX) &= \frac{(-1)^{N+r}}{2^{\alpha-r-1}} \mathcal{A}i_{\boldsymbol{\mu}}(X) + \mathcal{O}\left(\frac{1}{N^{1/3}}\right). \end{aligned}$$

Similarly, when $a_i = -1/2 + s_i/B$ for $i = 1, \dots, r$, we have

$$\begin{aligned} B^{1-i} e^{-(A+BX)/2} \widetilde{\Lambda}^{(i)}(A + BX) &= (-1)^{N+r} 2^{-\alpha-r} \widetilde{\text{Ai}}^{(i)}(X) + \mathcal{O}\left(\frac{1}{N^{1/3}}\right), \\ B^i e^{(A+BY)/2} \Lambda^{(i)}(A + BY) &= (-1)^{N+r} 2^{\alpha+r} \text{Ai}^{(i)}(Y) + \mathcal{O}\left(\frac{1}{N^{1/3}}\right). \end{aligned}$$

Proof. The proof is based on the saddle point method. For instance, consider

$$\widetilde{\Lambda}^{(i)}(A + BX) := \int_{\mathcal{C}_{\{0, a_1, \dots, a_r\}}} \frac{dz}{2\pi i} e^{Nf(z)} g(z)$$

with

$$f(z) = -4z + \ln(1+z) - \ln z, \quad g(z) = \frac{e^{-BXz} z^r (1+z)^{\alpha}}{\prod_{k=1}^i (z - a_k)}.$$

The function f has a simple saddle point of order two at $z_0 = -1/2$; that is,

$$f'(z_0) = 0 = f''(z_0) \quad f'''(z_0) \neq 0.$$

Since $f'''(z_0) = 32$, we have that $(z - z_0)f'''(z_0)$ is minimum when $\arg(z - z_0) = -\pi/3, \pi/3, \pi$. We choose $\pm\pi/3$. Doing the change of variable $v = B(z - z_0)$, we understand that the steepest descent contour in the complex v -plane starts at $\infty e^{-i\pi/3}$ crosses the origin and ends at $\infty e^{i\pi/3}$. Moreover,

$$Nf(z) = 2N - i\pi N + v^3/3 + O\left(N^{-1/3}\right),$$

while

$$g(z) = (-1)^r 2^{-r-\alpha} B^i e^{-Xv+BX/2} \prod_{k=1}^i (v - s_k)^{-1} \left(1 + O\left(N^{-1/3}\right)\right)$$

if $s_k = B(a_k + 1/2)$. Therefore,

$$\begin{aligned} \tilde{\Lambda}^{(i)}(A + BX) &= (-1)^{r+N} 2^{-r-\alpha} B^i e^{(A+BX)/2} \\ &\times \int_{\mathcal{A}_{\{s_1, \dots, s_i\}}} \frac{dv}{2\pi i} e^{-xv+v^3/3} \prod_{k=1}^i (v - s_k)^{-1} \left(1 + O\left(\frac{1}{N^{1/3}}\right)\right). \end{aligned}$$

This proves the asymptotics of $\tilde{\Lambda}^{(i)}$. The asymptotic behavior of the other functions is proved similarly. \square

Theorem 17. Suppose that Eq. (5.1) holds. Set $a_i = -1/2 + s_i/B$ for $i = 1, \dots, q$ and $a_i \gg -1/2$ for $i = q+1, \dots, r$. Following the notation of §3, let $\mathbf{s} := \{s_1, \dots, s_q\} = \{\sigma_i^{\mu_i}, \dots, \sigma_p^{\mu_q}\} =: \boldsymbol{\sigma}^\mu$ with $q = |\boldsymbol{\mu}|$. Moreover, let \mathcal{A}' be a contour going from $\infty e^{2i\pi/3}$ to $\infty e^{-2i\pi/3}$ and not intersecting \mathcal{A}_σ . Then, as $N \rightarrow \infty$,

$$BK_{\mathbf{m}}^\alpha(A + BX, A + BY) = \mathcal{K}_{\boldsymbol{\mu}}^{\text{soft}}(X, Y) + O\left(\frac{1}{N}\right),$$

where

$$\mathcal{K}_{\boldsymbol{\mu}}^{\text{soft}}(X, Y) = K^{\text{soft}}(X, Y) + \sum_{i=1}^q \tilde{\text{Ai}}^{(i)}(X) \text{Ai}^{(i)}(Y) \quad (6.13)$$

$$= \int_{\mathcal{A}_\sigma} \frac{du}{2\pi i} \int_{\mathcal{A}'} \frac{dv}{2\pi i} \frac{e^{-Xu+u^3/3} e^{Yv-v^3/3}}{v-u} \prod_{i=1}^p \left(\frac{v-\sigma_i}{u-\sigma_i}\right)^{\mu_i} \quad (6.14)$$

$$= \frac{1}{(X-Y)} \left(A(X, Y) + \sum_{i=1}^q \mu_i \tilde{\mathcal{A}}_{\boldsymbol{\mu}+\mathbf{e}_i}(X) \mathcal{A}_{\boldsymbol{\mu}-\mathbf{e}_i}(Y) \right) \quad (6.15)$$

with

$$A(X, Y) = \tilde{\mathcal{A}}_{\boldsymbol{\mu}}(X) \frac{d\mathcal{A}_{\boldsymbol{\mu}}(Y)}{dY} - \mathcal{A}_{\boldsymbol{\mu}}(Y) \frac{d\tilde{\mathcal{A}}_{\boldsymbol{\mu}}(X)}{dX}.$$

Proof. We first consider Eq. (6.13). The previous lemma implies

$$\tilde{\Lambda}^{(i)}(A + BX) \Lambda^{(i)}(A + BY) = B^{-1} \tilde{\text{Ai}}^{(i)}(X) \text{Ai}^{(i)}(Y) + O\left(\frac{1}{N^{2/3}}\right) \quad \text{when } i = 1, \dots, q.$$

For $i > q$, the supposition $a_i \gg -1/2$ is equivalent to $a_i = -1/2 + s_i/B$ together with $s_i = B\bar{s}_i$ for some positive real number \bar{s}_i . Then, Lemma 16 and Eqs (1.14)–(1.15) imply

$$\begin{aligned} \tilde{\Lambda}^{(i+q)}(A+BX)\Lambda^{(i+q)}(A+BY) \\ = B^{-2}\tilde{\text{Ai}}^{(q)}(X)\text{Ai}^{(q)}(Y) + \mathcal{O}\left(\frac{1}{N}\right) = \mathcal{O}\left(\frac{1}{N^{2/3}}\right) \quad \text{when } i = 1, \dots, r-q. \end{aligned}$$

Moreover, it is well known (see §2) that

$$BK_N^\alpha(A+BX, A+BY) = K^{\text{soft}}(X, Y) + \mathcal{O}\left(\frac{1}{N^{1/3}}\right).$$

From Proposition 11 we conclude that

$$\begin{aligned} BK_{\mathbf{m}}^\alpha(A+BX, A+BY) &= B\bar{K}_{\mathbf{m}}^\alpha(A+BX, A+BY) + \mathcal{O}\left(\frac{1}{N^{1/3}}\right) \\ &= BK_{N-r}^{\alpha+r}(A+BX, A+BY) + B \sum_{i=1}^r \tilde{\Lambda}^{(i)}(x)\Lambda^{(i)}(y) + \mathcal{O}\left(\frac{1}{N^{1/3}}\right) \\ &= K^{\text{soft}}(X, Y) + \sum_{i=1}^q \tilde{\text{Ai}}^{(i)}(X)\text{Ai}^{(i)}(Y) + \mathcal{O}\left(\frac{1}{N^{1/3}}\right) \end{aligned}$$

as desired.

We now prove Eq. (6.14). By using integration by parts, it is a simple exercise to show that the soft edge kernel defined in Eq. (1.14) can be written as a double integral,

$$K^{\text{soft}}(X, Y) = \int_{\mathcal{A}} \frac{du}{2\pi i} \int_{\mathcal{A}'} \frac{dv}{2\pi i} \frac{e^{-Xu+u^3/3} e^{Yv-v^3/3}}{v-u}.$$

Also, the integral representation of the incomplete multiple Airy functions, given in Eqs (1.15)–(1.16), leads to

$$\tilde{\text{Ai}}^{(i)}(X)\text{Ai}^{(i)}(Y) = \int_{\mathcal{A}_{\{s_1, \dots, s_i\}}} \frac{du}{2\pi i} \int_{\mathcal{A}'} \frac{dv}{2\pi i} e^{-Xu+u^3/3} e^{Yv-v^3/3} \frac{\prod_{k=1}^{i-1}(v-s_k)}{\prod_{k=1}^i(u-s_k)}.$$

Cauchy's theorem allows us to deform the contour $\mathcal{A}_{\{s_1, \dots, s_i\}}$ into $\mathcal{A}_{\{s_1, \dots, s_r\}} = \mathcal{A}_{\{\sigma_1, \dots, \sigma_d\}}$. Eq. (6.13) then gives

$$\mathcal{K}_{\boldsymbol{\mu}}^{\text{soft}}(X, Y) = \int_{\mathcal{A}_{\boldsymbol{\sigma}}} \frac{du}{2\pi i} \int_{\mathcal{A}'} \frac{dv}{2\pi i} e^{-Xu+u^3/3} e^{Yv-v^3/3} \left(\frac{1}{v-u} + \sum_{i=1}^q \frac{\prod_{k=1}^{i-1}(v-s_k)}{\prod_{k=1}^i(u-s_k)} \right).$$

Eq. (6.14) is then obtained by exploiting the decomposition (5.10).

We finally get Eq. (6.15) by multiplying Eq. (6.14) by $(X-Y)/(X-Y)$, which gives

$$\mathcal{K}_{\boldsymbol{\mu}}^{\text{soft}}(X, Y) = \frac{1}{X-Y} \int_{\mathcal{A}_{\boldsymbol{\sigma}}} \frac{du}{2\pi i} \int_{\mathcal{A}'} \frac{dv}{2\pi i} \frac{e^{u^3/3-v^3/3}}{v-u} \prod_{i=1}^p \left(\frac{v-\sigma_i}{u-\sigma_i} \right)^{\mu_i} \left(-\frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) (e^{-Xu+Yv}),$$

by integrating by parts, and by comparing the result with the definitions of the multiple Airy functions given in Eqs (6.10)–(6.11). \square

The ease at which the limiting kernel (6.13) can be deduced from Eq. (5.9) contrasts with the difficulty encountered in [4] in computing the same limiting kernel from a double contour integral representation equivalent to (5.2). The reason is that in the double contour the saddle point is such that $w = z$ in the integrand, but then the denominator vanishes. By having reduced the double contour to a sum of products of single integrals as in Eq. (5.9), the complication is avoided.

We finish this subsection by showing that the kernel of the perturbed Hermite ensemble can also be mapped to the soft edge kernel $\mathcal{K}_\mu^{\text{soft}}$. By doing this, we generalize a result previously obtained by P     [29] (i.e., Eq. (6.13) for $s_1 = \dots = s_r = 0$).

Proposition 18. *Let $\mathcal{K}_\mathbf{m}(x, y)$ be the kernel given in Proposition 12. Let also $A = \sqrt{2N}$ and $B = 1/\sqrt{2N^{1/3}}$. Suppose $a_k = \sqrt{2N}(-1 + s_k/N^{1/3})$ and $s_k > 0$ for $k = 1, \dots, q$. Suppose moreover $a_k \gg -A$ for $k = q+1, \dots, r$. Then, as $N \rightarrow \infty$,*

$$B\mathcal{K}_\mathbf{m}(A + BX, A + BY) = K^{\text{soft}}(X, Y) + \sum_{i=1}^q \widetilde{\text{Ai}}^{(i)}(X) \text{Ai}^{(i)}(Y) + \mathcal{O}(N^{-1/3}).$$

Proof. This formula is a straightforward consequence of Proposition 12 and the following asymptotic relations,

$$\begin{aligned} \widetilde{\Gamma}^{(i)}(A + BX) &= (-1)^{N+r+1} \frac{N^{(i-1)/3} e^{N/2}}{A^{N+i-r-1}} \widetilde{\text{Ai}}^{(i)}(X) \left(1 + \mathcal{O}(N^{-1/3})\right), \\ \Gamma^{(i)}(A + BX) &= (-1)^{N+r+1} \frac{A^{N+i-r} e^{-N/2}}{N^{i/3}} \text{Ai}^{(i)}(X) \left(1 + \mathcal{O}(N^{-1/3})\right). \end{aligned}$$

These equations can be obtained from the integral representation of the incomplete Hermite functions by using the steepest descent method (see for instance Lemma 16). We skip the detail. \square

6.3. From hard edge to soft edge. It is known (see for instance [15, §4]) that we can go from the hard edge to the soft by rescaling the spectral variables and taking the limit $\alpha \rightarrow \infty$:

$$2\alpha(\alpha/2)^{1/3} K^{\text{hard}}(X, Y) = K^{\text{soft}}(\xi, \eta) + \mathcal{O}(\alpha^{-1/3}) \quad (6.16)$$

if $X = \alpha^2 - (2\alpha^2)^{2/3}\xi$ and $Y = \alpha^2 - (2\alpha^2)^{2/3}\eta$. This formula is a consequence of the asymptotic relation between the Bessel and the Airy functions [28],

$$\left(\frac{\alpha}{2}\right)^{1/3} J_\alpha(\sqrt{X}) = \text{Ai}(\xi) + \mathcal{O}(\alpha^{-1/3}).$$

In the subsection, we show that the mapping between the hard and the soft edges is preserved in the perturbed Laguerre ensemble.

Lemma 19. *Let $X = \alpha^2 - (2\alpha^2)^{2/3}\xi$ and $\alpha\nu_i = 1/2 - \sigma_i/(4\alpha)^{1/3}$ with $\sigma_i > 0$ for $i = 1, \dots, d$. Then, as $\alpha \rightarrow \infty$,*

$$\begin{aligned} \left(\frac{\alpha}{2}\right)^{(1-|\mu|)/3} \widetilde{\mathcal{J}}_\mu^\alpha(\sqrt{X}|\nu) &= (-1)^{|\mu|} \widetilde{\mathcal{Ai}}_\mu(\xi|\sigma) + \mathcal{O}\left(\frac{1}{\alpha^{1/3}}\right) \\ \left(\frac{\alpha}{2}\right)^{(1+|\mu|)/3} \mathcal{J}_\mu^\alpha(\sqrt{X}|\nu) &= (-1)^{|\mu|} \mathcal{Ai}_\mu(\xi|\sigma) + \mathcal{O}\left(\frac{1}{\alpha^{1/3}}\right). \end{aligned}$$

Proof. The proof relies on saddle point method for both formulas. Here we only prove the first asymptotic development. We have from Eq. (6.2)

$$\tilde{\mathcal{J}}_{\boldsymbol{\mu}}^{\alpha}(\sqrt{X}|\boldsymbol{\nu}) = (2\alpha)^{\alpha} (1 - (\alpha/2)^{1/3} \xi/\alpha)^{\alpha} \int_{\mathcal{C}_{\{0\} \cup \boldsymbol{\nu}}} \frac{dz}{2\pi i} \frac{z^{|\boldsymbol{\mu}|+\alpha-1} e^{-\alpha^2 z + (2\alpha^2)^{2/3} \xi z + 1/(4z)}}{\prod_{i=1}^d (z - \nu_i)^{\mu_i}}$$

By making the change $z \mapsto z/\alpha$ and by considering $(1 - (\alpha/2)^{1/3} \xi/\alpha)^{\alpha} = e^{-(\alpha/2)^{1/3} \xi} + O(\alpha^{-1/3})$, we get

$$\tilde{\mathcal{J}}_{\boldsymbol{\mu}}^{\alpha}(\sqrt{X}|\boldsymbol{\nu}) = 2^{\alpha} e^{-(\alpha/2)^{1/3} \xi} \int_{\mathcal{C}_{\{0, \alpha\nu_1, \dots, \alpha\nu_d\}}} \frac{dz}{2\pi i} e^{\alpha f(z)} g(z) \left(1 + O(\alpha^{-1/3})\right),$$

where

$$f(z) = -z + 1/4z + \ln z \quad \text{and} \quad g(z) = \frac{e^{(4\alpha)^{1/3} \xi z} z^{|\boldsymbol{\mu}|-1}}{\prod_{i=1}^d (z - \alpha\nu_i)^{\mu_i}}.$$

The function $f(z)$ has a second order saddle point at $z_0 = 1/2$. We now suppose that $\alpha\nu_i < 1/2$ for all i . Following the method used in proving Proposition 16, we deform the closed contour $\mathcal{C}_{\{0, \alpha\nu_1, \dots, \alpha\nu_d\}}$ in such a way that it reaches the steepest descent contour, which is a path approaching z_0 from $\Im z < 0$ with an angle of $-2\pi/3$ and leaving z_0 for $\Im z > 0$ with an angle of $2\pi/3$. Setting

$$z = -v/(4\alpha)^{1/3} + 1/2 \quad \text{and} \quad \alpha\nu_i = 1/2 - \sigma_i/(4\alpha)^{1/3},$$

we find that

$$\begin{aligned} \alpha f(z) &= -2\alpha \ln 2 + v^3/3 + O(\alpha^{-1/3}) \\ g(z) &= \frac{(-1)^{|\boldsymbol{\mu}|} 2\alpha^{|\boldsymbol{\mu}|/3}}{2^{|\boldsymbol{\mu}|/3}} \frac{e^{-\xi v} e^{(\alpha/2)^{1/3} \xi}}{\prod_{i=1}^d (v - \sigma_i)^{\mu_i}} \left(1 + O(\alpha^{-1/3})\right). \end{aligned}$$

Returning to the integral representation of the type I multiple Bessel function, we get

$$\tilde{\mathcal{J}}_{\boldsymbol{\mu}}^{\alpha}(\sqrt{X}|\boldsymbol{\nu}) = (-1)^{|\boldsymbol{\mu}|} \left(\frac{\alpha}{2}\right)^{(|\boldsymbol{\mu}|-1)/3} \int_{\mathcal{A}_{\sigma}} \frac{dv}{2\pi i} \frac{e^{-\xi v + v^3/3}}{\prod_{k=1}^d (v - \sigma_k)^{\mu_k}} \left(1 + O(\alpha^{-1/3})\right)$$

Comparison with Eq. (6.10) shows that this is the expected result. \square

The above result can be applied to the incomplete Bessel and Airy functions. For instance, one can show from Eqs (6.4) and (6.12) that

$$2\alpha(\alpha/2)^{1/3} \left(\frac{X}{Y}\right)^{\alpha/2} \tilde{J}^{(i)}(X) J^{(i)}(Y) = \widetilde{\text{Ai}}^{(i)}(\xi) \text{Ai}^{(i)}(\eta) + O\left(\frac{1}{\alpha^{1/3}}\right).$$

When substituted into Theorem 15, this formula together with Theorem 17 imply the following asymptotics.

Proposition 20. *Let $X = \alpha^2 - (2\alpha^2)^{2/3} \xi$, $Y = \alpha^2 - (2\alpha^2)^{2/3} \eta$ and $\alpha\nu_i = 1/2 - \sigma_i/(4\alpha)^{1/3}$ for $\sigma_i > 0$. Then, as $\alpha \rightarrow \infty$,*

$$2\alpha(\alpha/2)^{1/3} \mathcal{K}_{\boldsymbol{\mu}}^{\text{hard}}(X, Y|\boldsymbol{\nu}) = \mathcal{K}_{\boldsymbol{\mu}}^{\text{soft}}(\xi, \eta|\boldsymbol{\sigma}) + O(\alpha^{-1/3}).$$

7. CONCLUDING REMARKS

7.1. The case $r = 1$ at the soft edge. Noting from (1.15) that $\widetilde{\text{Ai}}^{(i)}(x) \rightarrow 0$ as $x \rightarrow -\infty$, we see from (3.15) that

$$\widetilde{\text{Ai}}^{(1)}(X) = \int_{-\infty}^X e^{s_1(x-X)} \text{Ai}(x) dx.$$

Substituting in (1.12) shows that for $r = 1$, $s_1 = d$, $\mathcal{K}^{\text{soft}}(X, Y)$ can be written explicitly in terms of Airy functions

$$\mathcal{K}^{\text{soft}}(X, Y) = K^{\text{soft}}(X, Y) + \text{Ai}(Y) \int_{-\infty}^X e^{d(x-X)} \text{Ai}(x) dx. \quad (7.1)$$

In fact (7.1) has previously been derived as the soft edge scaled correlation kernel for random matrices closely related to (1.8) [16, 17]. These are $(N + 1) \times N$ complex Gaussian matrices \mathbf{X} with the entries of the first N rows independent complex Gaussians with mean 0 and variance $1/2$, while the entries of the final row are distributed according to this same complex Gaussian multiplied by \sqrt{c} . The quantities studied were the eigenvalues of $\mathbf{X}^\dagger \mathbf{X}$. With $c = 1/b$, such random matrices differ from those specified by (1.8), and the paragraph including (1.10) with $r = 1$, only in that the final row of \mathbf{X} has variance $1/2\sqrt{b}$ instead of the final column.

Results of [16, Prop. 14] give that with the largest n eigenvalues $\{x_j\}_{j=1,\dots,n}$ of $\mathbf{X}^\dagger \mathbf{X}$ scaled by

$$x_j \mapsto 4N + 2(2N)^{1/3} X_j \quad (7.2)$$

and with

$$b = \frac{1 + A}{2}, \quad A = \frac{d}{(2N)^{1/3}} \quad (7.3)$$

the corresponding n -point correlation function is given by

$$\det[\tilde{\mathcal{K}}^{\text{soft}}(X_j, X_k)]_{j,k=1,\dots,n} \quad (7.4)$$

where

$$\tilde{\mathcal{K}}^{\text{soft}}(X, Y) = -e^{d(X-Y)} \frac{\partial}{\partial X} \left(e^{-dX} \int_{-\infty}^Y e^{dv} K^{\text{soft}}(X, v) dv \right). \quad (7.5)$$

The scalings (7.2), (7.3) are identical to (1.10). Since we have shown that the soft edge scaling for the perturbed Laguerre ensemble is independent of the parameter α , one might suspect that it is independent of variance being altered along a row instead of down a column. Indeed use of the integral form of K^{soft} (1.14),

$$K^{\text{soft}}(X, Y) = \int_0^\infty \text{Ai}(X + t) \text{Ai}(Y + t) dt,$$

and use of integration by parts in (7.5) reproduces (7.1), but with $X \leftrightarrow Y$, an operation which leaves unaltered (7.4).

7.2. An open problem. For the same class of random matrices which gave rise to (7.5), the n -point correlation function for the smallest n eigenvalues $\{x_j\}_{j=1,\dots,n}$, scaled by

$$x_j \mapsto X_j/4N, \quad b = 2Nw$$

was computed in [16, Prop. 14] as being equal to

$$\det[\tilde{\mathcal{K}}^{\text{hard}}(X_j, X_k)]_{j,k=1,\dots,n}$$

where

$$\tilde{\mathcal{K}}^{\text{hard}}(X, Y) = -\frac{\partial}{\partial X} \left(e^{-wX/2} \int_0^Y e^{wv/2} K^{\text{hard}}(X, v) \Big|_{\alpha=0} dv \right). \quad (7.6)$$

We have not been able to express (7.6) in the form of (1.18) with $r = 1$. Thus unlike the situation at the soft edge, the eigenvalues at the hard edge distinguish the variance of the final row of \mathbf{X} being $1/\sqrt{b}$ rather than the variance of the final column.

This is not surprising when one recalls that for matrices $\mathbf{W}^\dagger \tilde{\mathbf{B}} \mathbf{W}$ with \mathbf{W} as in (1.8) and $\tilde{\mathbf{B}}$ an $n \times n$ positive definite matrix, the eigenvalues p.d.f. is given by [10, 30]

$$(-1)^{N(N-1)/2} \frac{\prod_{i=1}^n b_i^N \prod_{j < k}^N (\lambda_j - \lambda_k)}{\prod_{j=1}^N j! \prod_{j < k}^n (b_j - b_k)} \det \left[[b_j^{k-1}]_{\substack{j=1,\dots,n \\ k=1,\dots,N-n}} \quad [e^{-b_j \lambda_k}]_{\substack{j=1,\dots,n \\ k=1,\dots,N}} \right] \quad (7.7)$$

and is thus different in the neighbourhood of the smallest eigenvalues to (1.9). A challenge for future studies is to compute the n -point correlation for the hard edge scaling of the smallest eigenvalues specified by (7.7) with $\alpha = n - N \in \mathbb{Z}^+$ general.

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